THE WORLD ON A STRING

Abstract.

1. Introductory Remarks

The “World on a String” problem represents, in one simple-to-state problem, many of the difficulties encountered in computational science programs. In particular, it illustrates the following concepts:

1. Algebraic solutions are necessary but not sufficient for computing the answer.
2. Cancellation of significant digits is present in many different ways.
3. There are many different ways to do one thing but there is not just one way to do everything.
4. Methods for computing various standard problems may be pressed into service in funny ways.
5. Even physical constants, like the radius of the Earth, are subject to change. In the case at hand, various reference works may differ by ten percent in value.

2. The Problem Statement

Problem: Suppose we have a rope that is $l$ feet longer than the circumference of the Earth. How long a pole must we place under the rope to stretch it taut?

2.1. The Algebraic Solution. In order to solve this problem numerically, we first need to find a solution algebraically. Let $R$ be the radius of the Earth and $l$ represent the length of the excess. Then we have that the value of $h$ is the length of the pole (see Figure 1). Let $x$ be the length of the rope from the pole to the tangent point; this generates an angle $\theta$ between the point of tangency and the pole. Thus, we have $\tan \theta = x/R$. The rope is $2\pi R + l$ in length.

We can now see the right triangle $(AOB)$ which satisfies the Pythagorean relationship:

$$R^2 + x^2 = (R + h)^2.$$ 

Solving for $h$, we get

$$h = \sqrt{R^2 + x^2} - R.$$ 

To find $x$, we note that

$$2(\pi - \theta)R + 2x = circumference + l = rope,$$

and

$$2\pi R = circumference$$

Therefore, we get

$$-2R\theta + 2x = l.$$ 

Since $x = R \tan \theta$ we obtain

$$-2R\theta + 2R \tan \theta = l.$$
Or

\[ \tan \theta - \theta = \frac{l}{2R}. \]

While not particularly pleasant, this last equation gives us the (implicit) value of \( \theta \). This allows us to solve the problem, because \( x \), and thus \( h \), are determined.

Based on this formulation,

(1) Solve the equations, and

(2) Explain how many digits of the answer are justified.