Test 2 Solution, (100 points), CpSc 2070, Fall 2021

Question 1 (15 points): Use rules of inference [you cannot use truth table] to show that the three hypotheses (a) “If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on”, (b) “If the sailing race is held, then the trophy will be awarded”, and (c) “The trophy was not awarded” imply the conclusion “It rained”. You must provide precise justification for each of your steps.

Solution: Let
(a) p be the proposition “It rains.”
(b) q be the proposition “It is foggy.”
(c) r be the proposition “The sailing race will be held.”
(d) s be the proposition “The lifesaving demonstration will go on.”
(e) u be the proposition “The trophy will be awarded.”

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Question 2 (5 points): Prove that there is no positive integer $n$ such that: $90 < 2^n \times (n + 1) < 100$

Solution: Since $f(n) = 2^n \times (n + 1)$ is a strictly increasing function for all $n \geq 0$; it means that $f(n) < f(n + 1)$ for all $n \geq 0$.

Case 1: $n \geq 5$. Then $2^n \times (n + 1) \geq 2^5 \times (5 + 1) = 192 > 100$.

Case 2: $1 \leq n \leq 4$. Then $2^n \times (n + 1) \leq 2^4 \times (4 + 1) = 80 < 100$.

Since the above inequalities hold for both cases, it is true for all positive integer $n$.

Question 3 (10 points) Let $A$, $B$, $C$ be three sets. Show that

- a) $(A - C) \cap (C - B) = \emptyset$;
- b) $(B - A) \cup (C - A) = (B \cup C) - A$

You must provide precise justification for each of your steps.

Solution:

- a) $(A - C) \cap (C - B) = \emptyset$
  
  Suppose $x \in \text{LHS}$
  
  $\iff x \in (A - C)$ and $x \in (C - B)$ by definition of intersection
  
  $\iff (x \in A \land x \notin C)$ and $(x \in C \land x \notin B)$ by definition of set difference
  
  $\iff (x \notin C \land x \in C)$
  
  $\iff x \in C \cap C = \emptyset$ by definition of intersection and complement law
  
  $\iff \text{LHS}=\text{RHS}$.

- b) $(B - A) \cup (C - A) = (B \cup C) - A$
  
  Let $x \in \text{LHS}$
  
  $\iff x \in (B - A) \lor x \in (C - A)$ by definition of union
  
  $\iff (x \in B \land x \notin A) \lor (x \in C \land x \notin A)$ by definition of set difference
  
  $\iff (x \notin A) \land (x \in B \lor x \in C)$ by distributive law
  
  $\iff (x \in B \lor x \in C) \land (x \notin A)$ by commutative law
  
  $\iff (B \cup C) - A$ by definition of union and set difference
  
  $\iff x \in \text{RHS}$.

Question 4 (5 points): Calculate the number of binary strings of length 7, with the first two bits = 01, OR the last bit = 1. You must provide precise justification for each of your steps.

Solution: Let $A = \text{set of binary strings of length 7, with the first two bits = 01}$.

Let $B = \text{set of binary strings of length 7, with the last bit = 1}$.

We need to calculate: $|A \cup B|$. You must provide precise justification for each of your steps.
We have \(|A| = 2^{7-2} = 2^5 = 32; |B| = 2^{7-1} = 2^6 = 64; \) and \(|A \cap B| = 2^{7-3} = 2^4 = 16.\)
Thus: \(|A \cup B| = |A| + |B| - |A \cap B| = 32 + 64 - 16 = 80.\)

**Question 5 (8 points):** Determine whether each of these functions from \(Z\) to \(Z\) is one-to-one, onto. Circle either one, or both or none.

(a) \(f(n) = n - 1\) \hspace{1cm} one-to-one • onto • none
(b) \(f(n) = n^2 + 1\) \hspace{1cm} one-to-one x onto x none
(c) \(f(n) = n^3\) \hspace{1cm} one-to-one • onto x none
(d) \(f(n) = \lceil n/2 \rceil\) \hspace{1cm} one-to-one x onto • none

**Question 6 (10 points):** Determine whether each of these functions is a bijection from \(R\) to \(R\). You must provide precise justification for each of your steps.

(a) \(f(x) = -3x^2 + 7\)
(b) \(f(x) = (x+1)/(x+2) \) [This is a function from \(R\) to \(R - \{-2\}\)]
(c) \(f(x) = x^5 + 1\)

**Solution:**
(a) \(f(x) = -3x^2 + 7\) is not an one-to-one function since \(f(1) = 4 = f(-1)\).
(b) Strictly speaking, the function \(f\) is not even a function from \(R\) to \(R\), because \(f\) is undefined at \(-2\). If we consider the domain \(R' = R - \{-2\}\). Then \(f : R' \rightarrow R\) is one to one: for any \(x_1, x_2 \in R'\) such that \(f(x_1) \neq f(x_2)\), we need to show \(x_1 = x_2\). We have: \(f(x_1) = f(x_2) \iff (x_1+1)/(x_1+2) = (x_2+1)/(x_2+2) \iff x_1x_2 + 2x_1 + x_2 + 2 = x_1x_2 + x_1 + 2x_2 + 2 \iff x_1 = x_2\). But, \(f : R' \rightarrow R\) is not onto: because \(f(x) = (x+1)/(x+2) \neq 1\) for any \(x \in R'\). So 1 has no pre-image for \(f\). Thus, the function \(f\) is not a bijection.
(c) \(f(x) = x^5 + 1\) is one to one: for any \(x_1, x_2 \in R'\) such that \((x_1)^5 + 1 = (x_2)^5 + 1\), then we have \((x_1)^5 = (x_2)^5 \iff x_1 = x_2\). \(f(x) = x^5 + 1\) is onto: for any \(a \in R\), \(x^5 + 1 \iff x = (a - 1)^{1/5}\)

**Question 7 (8 points):** Find \(f \circ g\) and \(g \circ f\), where \(f(x) = x^2 + 1\) and \(g(x) = x + 2\), are functions from \(R\) to \(R\). Also, find \(f+g\) and \(fg\).

**Solution:** \(f \circ g = x^2 + 4x + 5\), \(g \circ f = x^2 + 3\), \((f + g)(x) = x^2 + x + 3\), \((fg)(x) = x^3 + 2x^2 + x + 2\)
Question 8 (8 points): List the first six terms of these sequences:
(a) the sequence whose $n^{th}$ term is the sum of the first $n$ positive integers
(b) the sequence whose $n^{th}$ term is $3^n - 2^n$
(c) the sequence whose $n^{th}$ term is $\sqrt[n]{n}$
(d) the sequence whose first two terms are 1 and 5 and each succeeding term is the sum of the two previous terms

Solution:
(a) 1, 3, 6, 10, 15, 21.
(b) 1, 5, 19, 65, 211, 665.
(c) 1, 1, 1, 2, 2, 2.
(d) 1, 5, 6, 11, 17, 28.

Question 9 (6 points): Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions:
(a) $a_n = 3a_{n-1}$, $a_0 = 1$; (b) $a_n = na_{n-1} + a^{2}_{n-2}$, $a_0 = -1$, $a_1 = 0$; (c) $a_n = a_{n-1} - a_{n-2} + a_{n-3}$, $a_0 = 1$, $a_1 = 1$, $a_2 = 2$.

Solution:
(a) $a_0 = 1$, $a_1 = 3 \cdot 1^2 = 3$; $a_2 = 3 \cdot 3^2 = 27$; $a_3 = 3 \cdot (3^2)^2 = 3^7$; $a_4 = 3 \cdot (3^3)^2 = 3^{15}$; $a_5 = 3 \cdot (3^{15})^2 = 3^{31}$.
(b) $a_0 = -1$, $a_1 = 0$; $a_2 = 2 \cdot 0 + 1^2 = 1$; $a_3 = 3 \cdot 1 + 0^2 = 3$; $a_4 = 4 \cdot 3 + 1^2 = 13$; $a_5 = 5 \cdot 13 + 3^2 = 74$.
(c) $a_0 = 1$; $a_1 = 1$; $a_2 = 2$; $a_3 = 2 - 1 + 1 = 2$; $a_4 = 2 - 2 + 1 = 1$; $a_5 = 1 - 2 + 2 = 1$.

Question 10 (10 points): Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n + 1)^2 = \frac{(n + 1)(2n + 1)(2n + 3)}{3}$ whenever $n$ is a nonnegative integer; use induction. You must provide precise justification for each of your steps.

Solution: We proceed by induction. The basis step, $n = 0$, is true, since $1^2 = (1 \cdot 1 \cdot 3)/3$. For the inductive step assume the inductive hypothesis that
$$1^2 + 3^2 + \cdots + (2k+1)^2 = \frac{(k + 1)(2k+1)(2k + 3)}{3}$$
We want to show that
$$1^2 + 3^2 + \cdots + (2k+1)^2 + (2k+3)^2 = \frac{(k + 2)(2k+3)(2k + 5)}{3}$$
(the right-hand side is the same formula with $k + 1$ plugged in for $n$). Now the left-hand side equals, by the inductive hypothesis, $(k + 1)(2k + 1)(2k + 3)/3 + (2k + 3)^2$. We need only do a bit
of algebraic manipulation to get this expression into the desired form: factor out \((2k + 3)/3\) and then factor the rest. In detail,

\[
(1^2 + 3^2 + 5^2 + \ldots +(2k+1)^2) + (2k + 3)^2
= (k + 1)(2k + 1)(2k + 3)/3 + (2k + 3)^2 \quad \text{(by the inductive hypothesis)}
= ((2k+3)/3) ((k+1)(2k+1) + 3(2k+3))
= ((2k+3)/3)((k+2)(2k+5))
= ((k+2)(2k+3)(2k+5))/3
\]

Question 11 (15 points): Given two positive real numbers \(x\) and \(y\), their **arithmetic mean** is \((x + y)/2\) and their **geometric mean** is \(\sqrt{xy}\). When we compare the arithmetic and geometric means of pairs of distinct positive real numbers, we find that the arithmetic mean is always greater than the geometric mean. [For example, when \(x = 4\) and \(y = 6\), we have \(5 = (4 + 6)/2 = 5 > \sqrt{4 \cdot 6} = \sqrt{24} < 5\).] Can you prove that this inequality is always true? You must provide precise justification for each of your steps.

**Solution:** To prove that \((x + y)/2 > \sqrt{xy}\) when \(x\) and \(y\) are distinct positive real numbers, we can work backward. We construct a sequence of equivalent inequalities. The equivalent inequalities are \((x + y)/2 > \sqrt{xy}\), or \((x + y)^2 > 4xy\), or \((x + y)^2 > 4xy\), \(x^2 + 2xy + y^2 > 0\), \((x - y)^2 > 0\).

Because \((x - y)^2 > 0\) when \(x \neq y\), it follows that the final inequality is true. Because all these inequalities are equivalent, it follows that \((x + y)/2 > \sqrt{xy}\) when \(x \neq y\).

Once we have carried out this backward reasoning, we can easily reverse the steps to construct a proof using forward reasoning. We now give this proof.

Suppose that \(x\) and \(y\) are distinct positive real numbers. Then \((x - y)^2 > 0\) because the square of a nonzero real number is positive. Because \((x - y)^2 = x^2 - 2xy + y^2\), this implies that \(x^2 - 2xy + y^2 > 0\). Adding \(4xy\) to both sides, we obtain

\[x^2 + 2xy + y^2 > 4xy.\]

Because \(x^2 + 2xy + y^2 = (x + y)^2\), this means that \((x + y)^2 \geq 4xy\).

Dividing both sides of this equation by 4, we see that \((x + y)^2/4 > xy\).

Finally, taking square roots of both sides (which preserves the inequality because both sides are positive) yields \((x + y)/2 > \sqrt{xy}\).

We conclude that if \(x\) and \(y\) are distinct positive real numbers, then their arithmetic mean \((x + y)/2\) is greater than their geometric mean \(\sqrt{xy}\).
**Bonus Question (10 points):** Prove that \(3+3 \cdot 5+3 \cdot 5^2+\cdots+3 \cdot 5^n = 3(5^{n+1} - 1)/4\) whenever \(n\) is a nonnegative integer; use induction. You must provide precise justification for each of your steps.

Let \(P(n)\) be the proposition \(3 + 3 \cdot 5 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^n = 3(5^{n+1} - 1)/4\). To prove that this is true for all nonnegative integers \(n\), we proceed by mathematical induction. First we verify \(P(0)\), namely that \(3 = 3(5 - 1)/4\), which is certainly true. Next we assume that \(P(k)\) is true and try to derive \(P(k+1)\). Now \(P(k+1)\) is the formula

\[
3 + 3 \cdot 5 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^k + 3 \cdot 5^{k+1} = \frac{3(5^{k+2} - 1)}{4}.
\]

All but the last term of the left-hand side of this equation is exactly the left-hand side of \(P(k)\), so by the inductive hypothesis, it equals \(3(5^{k+1} - 1)/4\). Thus we have

\[
3 + 3 \cdot 5 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^k + 3 \cdot 5^{k+1} = \frac{3(5^{k+1} - 1)}{4} + 3 \cdot 5^{k+1}
\]

\[
= 5^{k+1} \left(\frac{3}{4} + 3\right) - \frac{3}{4} = 5^{k+1} \cdot \frac{15}{4} - \frac{3}{4}
\]

\[
= 5^{k+2} \cdot \frac{3}{4} - \frac{3}{4} = \frac{3(5^{k+2} - 1)}{4}.
\]