1 (10 points). For the given collections of premises, you need to draw the conclusion that “It rained on Thursday”. Explain the rules of inference used to obtain each conclusion from the premises (givens): (a) “If I take the day off, it either rains or snows.”, (b) “I took Tuesday off or I took Thursday off.”, (c) “It was sunny on Tuesday.”, and (d) “It did not snow on Thursday.”

Solution:
Let
• O (x) be “I take the day x off.”
• R (x) be “It rains on day x.”
• S (x) be “It snows on day x.”

Then the premises are:

• “If I take the day off, it either rains or snows.” leads to \( \forall x \ (O(x) \rightarrow R(x) \lor S(x)) \)
• “I took Tue off or I took Thr off” leads to \( O(Tue) \lor O(Thr) \)
• “It was sunny on Tuesday” leads to \( \neg R \ (Tue) \land \neg S \ (Tue). \)
• “It did not snow on Thr” leads to \( \neg S(Thr) \)

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
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<tbody>
<tr>
<td>1. ( \forall x \ (O(x) \rightarrow R(x) \lor S(x)) )</td>
<td>Premise (a)</td>
</tr>
<tr>
<td>2. ( O(Tue) \lor O(Thr) )</td>
<td>Premise (b)</td>
</tr>
<tr>
<td>3. ( \neg R \ (Tue) \land \neg S \ (Tue) )</td>
<td>Premise (c)</td>
</tr>
<tr>
<td>4. ( S \ (Thr) )</td>
<td>Premise (d)</td>
</tr>
<tr>
<td>5. ( \neg (R \ (Tue) \lor S \ (Tue)) )</td>
<td>De Morgan’s laws on (3)</td>
</tr>
<tr>
<td>6. ( O \ (Tue) \rightarrow R \ (Tue) \lor S \ (Tue) )</td>
<td>Universal instantiation from (1)</td>
</tr>
<tr>
<td>7. ( \neg O \ (Tue) )</td>
<td>from (5) and (6) [Using ( \neg p \land (q \rightarrow \neg p) \equiv \neg p \land (\neg q \lor p) ) ]</td>
</tr>
<tr>
<td>8. ( O \ (Thr) )</td>
<td>from (2) and (7)</td>
</tr>
<tr>
<td>9. ( O \ (Thr) \rightarrow R \ (Thr) \lor S \ (Thr) )</td>
<td>Universal instantiation from (1)</td>
</tr>
<tr>
<td>10. ( R \ (Thr) \lor S \ (Thr) )</td>
<td>from (8) and (9)</td>
</tr>
<tr>
<td>11. ( R \ (Thr) )</td>
<td>from (4) and (10)</td>
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</table>

Therefore, “It rained on Thursday”. Conclusion
2 (20 points). Show that if A, B and C are sets, then \( A \cap B \cap C = \overline{A} \cup \overline{B} \cup \overline{C} \) by showing each side is a subset of the other side (provide 2 proofs one using logical arguments and the other using truth table.

FIRST PART \( \text{Let } x \in A \cap B \cap C. \)

Using the definition of the complement, \( x \) is in the complement of \( A \cap B \cap C \) when \( x \) is not in \( A \cap B \cap C \)

\[-(x \in A \cap B \cap C)\]

Using the definition of the intersection, \( x \) is in the intersection of two sets when it is both sets.

\[-(x \in A \land x \in B \land x \in C)\]

Using De Morgan's law for propositions:

\[-(x \in A) \lor -(x \in B) \lor -(x \in C)\]

Using the definition of the complement, \( x \) is in the complement of a set when \( x \) is not in the set.

\[x \in \overline{A} \lor x \in \overline{B} \lor x \in \overline{C}\]

Using the definition of the union, \( x \) is in the union of two sets when it is either set.

\[x \in \overline{A} \cup \overline{B} \cup \overline{C}\]

By the definition of a subset, we have then shown \( A \cap B \cap C \subseteq \overline{A} \cup \overline{B} \cup \overline{C} \).

(b) If \( x \) is an element, then 1 represents that the element is in the set and 0 represents that the element is not in the set.

\[
\begin{array}{cccccccc}
A & B & C & A \cap B \cap C & \overline{A} & \overline{B} & \overline{C} & \overline{A} \cup \overline{B} \cup \overline{C} \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Since the last two columns have the same values, the two expressions are equal.

\[A \cap B \cap C = \overline{A} \cup \overline{B} \cup \overline{C}\]
3 (10 points). Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) the negative integers
b) the even integers
c) the integers less than 100
d) the real numbers between 0 and 1/2
e) the positive integers less than 1,000,000,000
f) the integers that are multiples of 7

Solution:
(a) Countably infinite
(b) Countably infinite
(c) Countably infinite
(d) Uncountable
(e) Finite
(f) Countably infinite

4 (10 points). Show that if A and B are sets and \( A \subset B \) then \(|A| \leq |B|\).

Solution:

By the definition of a subset: if \( a \in A \), then \( a \in B \). Define a function \( f \) as: \( f : A \rightarrow B \), \( f(a) = a \). We need to check the function \( f \) is one-to-one. Let \( f(a) = f(b) \). By definition of \( f \), we get \( a = b \), and thus \( f \) is one-to-one. Since \( f \) is one-to-one, \(|A| \leq |B|\) [we know that if there is a one-to-one from \( A \) to \( B \), \(|A| \leq |B|\), by definition]