Mathematical Induction

Rosen, 6th ed. 4.1, 4.2
Suppose we have a ladder of \( n \) rungs. Let’s say we can guarantee two things:

- We can reach the first rung of the ladder.
- If we can reach the \( i^{th} \) rung of the ladder, then we can reach the next (i.e., the \( (i + 1)^{st} \)) rung.

What can we conclude, then?

We can conclude that we can reach the \( n^{th} \) rung for any \( n \).
\textbf{Basics}

\textbf{The Well-Ordering Property} - Every nonempty set of nonnegative integers has a least element.

Many theorems state that \( P(n) \) is true for all positive integers.

\begin{itemize}
  \item For example, \( P(n) \) could be the statement that the sum of the first \( n \) positive integers \( 1+2+3+\ldots+n = \frac{n(n+1)}{2} \)
  \item \textbf{Mathematical Induction} is a technique for proving theorems of this kind.
\end{itemize}
Steps in an Induction Proof to show that $P(n)$ is true.

Goal: to prove $P(n)$ is true (where $n$ is a positive integer).

1. **Basis step**: The proposition is shown to be true for $n=1$ (or, more generally, the first element in the set)

2. **Inductive step**: The implication $P(n) \implies P(n+1)$ is shown to be true for every positive integer $n$.
   
   For $n \in \mathbb{Z}^+$, $[P(1) \land \forall n (P(n) \implies P(n+1))] \implies \forall n P(n)$

**Note**: Here $P(k)$ is called the inductive assumption (or inductive hypothesis).

**Note**: Both steps are necessary.
**Example 1: Show** \(2^n > n\)

**Proof:** **Basis step:** When \(n = 1\), we have \(2n = 2 > 1 = n\). So the proposition is true for \(n = 1\).

**Inductive step:** Assume that the proposition is true for \(n = k\) (where \(k\) is a positive integer), i.e., \(2^k > k\).

Now we prove that it is also true for \(n = k + 1\), i.e., \(2^{k+1} > k + 1\). From \(2^k > k\) we get that \(2^{k+1} = 2 \times 2^k > 2 \cdot k \geq k + 1\).

This completes the induction proof.
Understanding first example

In the first example, we have shown two things:

(a) $2^1 > 1$;

(b) If $2^k > k$ for positive integer $k$, then $2^{k+1} > k + 1$.

Hence, we have the following statements being true:

(1) $2^1 > 1$; (This is (a))

(2) If $2^1 > 1$, then $2^2 > 2$; (This is (b) when $k = 1$)

(3) If $2^2 > 2$, then $2^3 > 3$; (This is (b) when $k = 2$)

... 

(n) If $2^{n-1} > n - 1$, then $2^n > n$; (This is (b) when $k = n - 1$)

Putting all of them together, we see that $2^n > n$. 
Example 2: Show that $3 \mid n^3 - n$ for positive integer $n$.

Proof: Basis step: When $n = 1$, we have $n^3 - n = 0$. Clearly, $3 \mid n^3 - n$.

Inductive step: Assume that $3 \mid k^3 - k$ for positive integer $k$. We’ll show that $3 \mid (k + 1)^3 - (k + 1)$.

It is easy to see $(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 2k = (k^3 - k) + 3(k^2 + k)$.

Since $3 \mid k^3 - k$, we can write $k^3 - k = 3j$ where $j$ is an integer.

So,

$$(k + 1)^3 - (k + 1) = 3j + 3(k^2 + k) = 3(j + k^2 + k)$$

Hence, $3 \mid (k + 1)^3 - (k + 1)$. 
Understanding second example

In the second example, we have shown two things:

(a) $3|1^3 - 1$;
(b) If $3|(k^3 - k)$ for positive integer $k$, then $3|\{(k + 1)^3 - (k + 1)\}$.

Hence, we have the following statements being true:

(1) $3|1^3 - 1$; (This is (a))
(2) If $3|1^3 - 1$, then $3|2^3 - 2$; (This is (b) when $k = 1$)
(3) If $3|2^3 - 2$, then $3|3^3 - 3$; (This is (b) when $k = 2$)

... 
(n) If $3|(n - 1)^3 - (n - 1)$, then $3|n^3 - n$; (This is (b) when $k = n - 1$)

Putting all of them together, we see that $3|n^3 - n$. 
Example: If \( p(n) \) is the proposition that the sum of the first \( n \) positive integers is \( n(n+1)/2 \), prove \( p(n) \) for \( n \in \mathbb{Z}^+ \).

Prove that \( \sum_{j=1}^{n} j = \frac{n(n+1)}{2} \)

**Basis Step**: Show that \( p(1) \) is true.

\[
\sum_{j=1}^{1} j = 1
\]

\[
\frac{1(1+1)}{2} = \frac{2}{2} = 1
\]
**Inductive Step: Show that** \( p(n) \rightarrow p(n+1) \)

We want to show that

\[
\left[ \sum_{j=1}^{n} j = \frac{n(n+1)}{2} \right] \rightarrow \left[ \sum_{j=1}^{n+1} j = \frac{(n+1)(n+2)}{2} \right]
\]
is true.

Assume that

\[
\left[ \sum_{j=1}^{n} j = \frac{n(n+1)}{2} \right]
\]

Then

\[
\sum_{j=1}^{n+1} j = \left[ \sum_{j=1}^{n} j \right] + (n+1) = \frac{n(n+1)}{2} + n+1 = \frac{n(n+1)}{2} + \frac{2(n+1)}{2}
\]

\[
= \frac{n^2 + n + 2n + 2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}
\]
Inductive Step: Show that \( p(n) \rightarrow p(n+1) \)

For this problem we want to show that

\[
\left[ \sum_{j=1}^{n} j = \frac{n(n+1)}{2} \right] \rightarrow \left[ \sum_{j=1}^{n+1} j = \frac{(n+1)(n+2)}{2} \right]
\]

is true.

Assume that

\[
\sum_{j=1}^{n} j = \frac{n(n+1)}{2}
\]

Then

\[
\sum_{j=1}^{n+1} j = \left[ \sum_{j=1}^{n} j \right] + (n+1) = \frac{n(n+1)}{2} + n + 1 = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{n^2 + 2n + 2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}
\]

Why \((n+1)\) and \((n+2)\)?

Why can we assume this?

Same sum, different format

What we wanted to show.
What have we shown?

\[
\sum_{k=1}^{n} k = \frac{n(n+1)}{2}
\]

works for \(n = 1\).

And if it works for \(n = 1\), then it works for \(n = 2\) and if it works for \(n = 2\), then it works for \(n = 3\) and if it works for \(n = 3\), then it work for \(n = 4\) etc.

So the closed form solution must work for all integers > 0.
Variants of Mathematical Induction

In the mathematical induction we just studied, the constraint is that $n$ is a positive integer. In fact, we can have variants:

- $n$ is a non-negative integer;
- or, $n$ is a positive integer $\geq m$.

To deal with the above situations, all we need is:

- adjust the basis step, so that it considers $n = 0$ or $n = m$ instead of $n = 1$.
- adjust the inductive step, so that $P(k) \rightarrow P(k + 1)$ is proved for all non-negative integer $k$ or all integer $k \geq m$. 
Example for variant

Suppose that, for a finite set $S$, $|S| = n$. Show that $|P(S)| = 2^n$.

- Note that we cannot consider $n = 1$ in the basis step! Because $S$ could be the empty set and thus $n$ could be 0.

- That means, we have to make sure the above statement is true for all non-negative integer $n$ (not just all positive integer $n$).

- If we consider $n = 1$ in the basis step, then the entire proof ignores the possibility of $n = 0$.

- Similarly, when we do the inductive step, we cannot just prove it for all positive integer $k$. We should prove it for all non-negative integer $k$. 
**Example for variant**

**Proof:** Basis step: When \( n = 0 \), \( S \) is the empty set. Hence, \( P(S) = \{\phi\} \), which means \( |P(S)| = 1 = 2^0 \).

**Inductive step:** Assume that, for all \( S \) such that \( |S| = k \) (where \( k \) is a non-negative integer), \( |P(S)| = 2^k \).

Now we show that, for all \( S' \) such that \( |S'| = k + 1 \), \( |P(S')| = 2^{k+1} \).

Clearly, all \( S' \) such that \( |S'| = k + 1 \) can be written as \( S' = S \cup \{a\} \), where \( |S| = k \) and \( a \) is not in \( S \).

To count \( |P(S')| \), i.e., the number of subsets of \( S' \), we only need to count:

- (a) \( |P(S)| \), i.e., the number of subsets of \( S \); By the inductive assumption, we know that \( |P(S)| = 2^k \).
- (b) The number of subsets of \( S' \) that contains \( a \).

We note that each subset containing \( a \) uniquely corresponds to a subset not containing \( a \) (by eliminating \( a \) from the subset).

Hence, this number is also \( |P(S)| = 2^k \).

We sum up these two numbers and get that \( |P(S')| = 2^k + 2^k = 2^{k+1} \).
**Prove:** If \( p(n) \) is the proposition that the sum of the first \( n \) odd integers is \( n^2 \), prove \( p(n) \) for \( n \in \mathbb{Z}^+ \)

Let’s first look at an example so we can figure out what we are trying to prove! Try first 5 odd integers.

\[ 1 + 3 + 5 + 7 + 9 = 5^2 \]

If \( a_1, a_2, a_3 \), are all odd integers, then what does a general term look like?

\[ 2k - 1, \text{ for } k = 1, 2, 3, \ldots, n \]

So we want to prove that \( \sum_{k=1}^{n} (2k - 1) = n^2 \)
Prove that \[ \sum_{k=1}^{n} (2k - 1) = n^2 \]

Basis Step: \[ \sum_{k=1}^{1} (2k - 1) = (2 \times 1 - 1) = 1 = 1^2 \]

Inductive Step: Show that \[ \sum_{k=1}^{n} (2k - 1) = n^2 \rightarrow \sum_{k=1}^{n+1} (2k - 1) = (n + 1)^2 \]

Assume that \[ \sum_{k=1}^{n} (2k - 1) = n^2 \]

\[ \sum_{k=1}^{n+1} (2k - 1) = \left[ \sum_{k=1}^{n} (2k - 1) \right] + (2(n + 1) - 1) = n^2 + 2n + 1 = (n + 1)^2 \]
If \( p(n) \) is the proposition that prove \( p(n) \) when \( n \) is a non-negative integer.

\[
\sum_{j=0}^{n} 2^j = 2^{n+1} - 1
\]

**Basis Step:** We will show \( p(0) \) is true. \[
\sum_{j=0}^{0} 2^j = 2^0 = 1; 2^{0+1} - 1 = 2 - 1 = 1
\]

**Inductive Step:** We want to show that \( p(n) \rightarrow p(n+1) \) i.e.,

\[
\sum_{j=0}^{n} 2^j = 2^{n+1} - 1 \rightarrow \sum_{j=0}^{n+1} 2^j = 2^{n+2} - 1
\]

Assume \( \sum_{j=0}^{n} 2^j = 2^{n+1} - 1 \) then,

\[
\sum_{j=0}^{n+1} 2^j = \left[ \sum_{j=0}^{n} 2^j \right] + 2^{n+1} = 2^{n+1} - 1 + 2^{n+1} = 2 \times 2^{n+1} - 1 = 2^{n+2} - 1
\]
Can we use induction to prove that \[ \sum_{j=1}^{n} j = \frac{n^2}{2} + \frac{n}{2} + 1 \]

**Induction Step:**
Assume the closed form solution is true for \( n \). We must show that
\[ \sum_{j=1}^{n} j = \frac{n^2}{2} + \frac{n}{2} + 1 \rightarrow \sum_{j=1}^{n+1} j = \frac{(n+1)^2}{2} + \frac{n+1}{2} + 1 \]

\[ \sum_{j=1}^{n+1} j = \frac{(n+1)^2}{2} + \frac{n+1}{2} + 1 = \left( \sum_{j=1}^{n} j \right) + n + 1 = \frac{n^2}{2} + \frac{n}{2} + 1 + n + 1 \]

\[ = \frac{n^2}{2} + \frac{n}{2} + 1 + \frac{2n + 2}{2} = \frac{n^2}{2} + \frac{2n + 1}{2} + \frac{n + 1}{2} + 1 \]

The induction step works! But is this a valid closed form solution?
Basis Step:

\[ \sum_{j=1}^{1} j = 1 \] \[ \frac{1^2}{2} + \frac{1}{2} + 1 = 2 \] but, \( 1 \neq 2 \)

Since the basis step does not work, the closed form solution is not valid!

An induction proof is a valid proof \textbf{only if} the basis step AND the induction step are both true. Neither step by itself proves anything.
Let \( p(n) \) be the statement that \( n! > 2^n \). Prove \( p(n) \) for \( n \geq 4 \).

**Inductive Proof:**

**Basis Step:** We will show that \( p(4) \) is true.

\[
4! = 24 > 2^4 = 16
\]

**Inductive Step:** We want to show that \( \forall k \geq 4, p(k) \rightarrow p(k+1) \). Assume \( k! > 2^k \) for some arbitrary \( k \geq 4 \).
\[
\begin{align*}
(n+1)! &= (k+1)k! \\
&> (k+1)2^k \text{ (inductive hypothesis)} \\
&> 2*2^k \text{ (since } k \geq 4) \\
&= 2^{k+1}
\end{align*}
\]

Since \( p(4) \) is true and \( p(n) \to p(n+1) \), then \( p(n) \) is true for all integers \( n \geq 4 \).
Let $p(n)$ be the statement that all numbers of the form $8^n - 2^n$ for $n \in \mathbb{Z}^+$ are divisible by 6 (i.e., can be written as $6k$ for some $k \in \mathbb{Z}$). Prove $p(n)$

**Inductive Proof**

**Basis Step:** We will show that $p(1)$ is true.

$8^1 - 2^1 = 6$ which is clearly divisible by 6.

**Inductive Step:** We must show that $\forall k \in \mathbb{Z}^+ ((8^k - 2^k)$ is divisible by 6 $\rightarrow (8^{k+1} - 2^{k+1})$ is divisible by 6.
Divisible by 6 Example (cont.)

\[ 8^{k+1} - 2^{k+1} = 8(8^k) - 2^{k+1} \]
\[ = 8(8^k) - 8(2^k) + 8*2^k - 2^{k+1} \]
\[ = 8(8^k-2^k) + 8*2^k - 2^{k+1} \]
\[ = 8(8^k-2^k) + 8*2^k - 2*2^k \]
\[ = 8(8^k-2^k) + 6*2^k \]

By the inductive hypothesis \( 8(8^k-2^k) \) is divisible by 6 and clearly \( 6*2^k \) is divisible by 6. Thus \( 8^{k+1} - 2^{k+1} \) is divisible by 6. Since \( p(1) \) is true and \( p(n) \rightarrow p(n+1) \), then \( p(n) \) is true for all nonnegative integers \( n \).
Prove that 21 divides $4^{n+1} + 5^{2n-1}$ whenever $n$ is a positive integer

Basis Step: When $n = 1$, then $4^{n+1} + 5^{2n-1} = 4^{1+1} + 5^{2(1)-1} = 4^2 + 5 = 21$ which is clearly divisible by 21.

Inductive Step: Assume that $4^{n+1} + 5^{2n-1}$ is divisible by 21. We must show that $4^{n+1+1} + 5^{2(n+1)-1}$ is divisible by 21.
Continued from the last page

\[ 4^{n+1} + 5^{2(n+1)-1} = 4 \cdot 4^{n+1} + 5^{2n+2-1} \]
\[ = 4 \cdot 4^{n+1} + 25 \cdot 5^{2n-1} \]
\[ = 4 \cdot 4^{n+1} + (4 + 21) \cdot 5^{2n-1} \]
\[ = 4(4^{n+1} + 5^{2n-1}) + 21 \cdot 5^{2n-1} \]

The first term is divisible by 21 by the induction hypothesis and clearly the second term is divisible by 21. Therefore their sum is divisible by 21.
Second Principle of Mathematical Induction (Strong Induction)

1. **Basis Step:** The proposition \( p(1) \) is shown to be true.

2. **Inductive Step:** It is shown that \([p(1) \land p(2) \land \ldots \land p(n)] \rightarrow p(n+1)\) is true for every positive integer \( n \).

3. Sometimes we have *multiple* basis steps to prove.
Consider the sequence $a_1, a_2, a_3, \ldots$ defined as $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. Prove that $a_n < 2^n$.

**Basis Step:**

$$a_4 = a_1 + a_2 + a_3 = 1 + 2 + 3 = 6 < 2^4 = 16.$$  

**Inductive Step:**

Show that $(a_i < 2^i$ for $i \in \mathbb{Z}^+$ and $i < k) \rightarrow a_k < 2^k$

$$a_k = a_{k-1} + a_{k-2} + a_{k-3} < 2^{k-1} + 2^{k-2} + 2^{k-3}$$

$$< (2^2 + 2 + 1)2^{k-3} = 7(2^{k-3}) < 8(2^{k-3}) = 2^3(2^{k-3}) = 2^k.$$  

Therefore $a_k < 2^k$
Consider the sequence $a_1$, $a_2$, $a_3$, ... defined as $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. Prove that $a_n < 2^n$.

**Basis Step:**

\[ a_4 = a_1 + a_2 + a_3 = 1 + 2 + 3 = 6 < 2^4 = 16. \]

**Inductive Step:**

Show that $(a_i < 2^i$ for $i \in \mathbb{Z}^+$ and $i < k) \rightarrow a_k < 2^k$

\[ a_k = a_{k-1} + a_{k-2} + a_{k-3} < 2^{k-1} + 2^{k-2} + 2^{k-3} \]
\[ < (2^2 + 2 + 1)2^{k-3} = 7(2^{k-3}) < 8(2^{k-3}) = 2^3(2^{k-3}) = 2^k. \]

Therefore $a_k < 2^k$
Example of Strong Induction

Consider the sequence defined as follows:

\[ b_0 = 1, \ b_1 = 1 \]

\[ b_n = 2b_{n-1} + b_{n-2} \text{ for } n>1 \]

1,1,3,7,17,…

Prove that \( b_n \) is always odd
Inductive Proof Using Strong Induction

Basis Cases: (one for n=0 and one for n=1 since the general formula is not applicable until n>1.)

\[ b_0 = b_1 = 1 \] so both \( b_0 \) and \( b_1 \) are odd.

Inductive Step:
Consider \( k>1 \) and assume that \( b_n \) is odd for all \( 0 \leq n \leq k \). We must show that \( b_{k+1} \) is odd.
From the formula we know that

$$b_{k+1} = 2b_k + b_{k-1}.$$ Clearly the first term is even. By the inductive hypothesis the second term is odd. Since the sum of an even integer and an odd integer is always odd (which we proved in number theory), then $b_{k+1}$ is odd.

In this example we did not need all $p(n)$, $0 \leq n \leq k$, but we did need $p(k-1)$. Note that a proof using weak induction would only be able to assume $p(k)$. 

Proof Example (cont.)
Another Example for strong induction

Show that any positive integer $n > 1$ can be written as the product of primes.

Note this is actually part of the fundamental theorem of arithmetic. Here we prove it using strong induction.

Proof: Basis step: Here we consider $n = 2$ in stead of $n = 1$, because there is a restriction $n > 1$.

When $n = 2$, since 2 is by itself a prime, the proposition is clearly true.
Inductive step: Assume every $n$ such that $1 < n \leq k$ (where $k$ is an integer $> 1$) can be written as the product of primes.

Now we show that $k + 1$ can also be written as the product of primes. We consider two cases:

- Case A: $k + 1$ is a prime. Then we are done.
- Case B: $k + 1$ is a composite. Then there exist positive integers $a > 1$ and $b > 1$ such that $k + 1 = a \cdot b$. Since $a > 1$, we know $a \geq 2$, and thus $b \leq (k + 1)/2 < k$. [why?]

By the inductive assumption, $b$ can be written as the product of primes.

Similarly, $a$ can also be written as the product of primes.

Combining these two results, we see that $k + 1 = a \cdot b$ can be written as the product of primes.
Understanding the previous example

In this example, we have shown two things:

(a) 2 can be written as the product of primes;
(b) If all n such that \(1 < n \leq k\) can be written as the product of primes, then \(k + 1\) can be written as the product of primes.

Hence, we have the following statements being true:

(1) 2 can be written as the product of primes; (This is (a))
(2) If 2 can be written as the product of primes, then 3 can be written as the product of primes; (This is (b) when \(k = 2\))
(3) If 2 and 3 can be written as the product of primes, then 4 can be written as the product of primes; (This is (b) when \(k = 3\)) . . .
(n-1) If 2, 3, . . . , and \(n − 1\) can be written as the product of primes, then \(n\) can be written as the product of primes; (This is (b) when \(k = n − 1\))

Putting all of them together, we see that \(n\) can be written as the product of primes.
Harmonic Numbers

An Inequality for **Harmonic Numbers**: The harmonic numbers $H_j$, $j = 1, 2, 3, \ldots$, are defined by

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{j}, \text{ Example: } H_4 = \frac{25}{12}$$

Prove that

$$H_{2^n} \geq 1 + \frac{n}{2} \quad \text{when } n \text{ is a nonnegative integer.}$$

In order to prove this, let $P_n$ be the proposition $H_{2^n} \geq 1 + \frac{n}{2}$

Basis Step: $P(0)$ is true since $H_1 = 1 \geq 1 + 0/2 \quad [2^0 = 1]$

For the sake of simplicity of typing, let’s say $N(k) = 2^k$, for any nonnegative integer $k$. 
**INDUCTIVE STEP**: The inductive hypothesis is the statement that $P(k)$ is true, that is, $H_{N(k)} \geq 1 + k/2$. Where $k$ is an arbitrary nonnegative integer. We must show that if $P(k)$ is true then $P(k+1)$, i.e., $H_{N(k+1)} \geq 1 + (k+1)/2$.

$H_{N(k+1)} = 1 + 1/2 + 1/3 + 1/N(k) + \ldots + 1/N(k) + 1/{N(k)+1} + \ldots + 1/N(k+1)$

$= H_{N(k)} + 1/{N(k)+1} + \ldots + 1/N(k+1)$

$\geq 1 + k/2 + 1/{N(k)+1} + \ldots + 1/N(k+1)$ [inductive hypothesis]

$\geq 1 + k/2 + N(k).1/N(k+1)$ [because there are $N(k)$ terms each $\geq 1/N(k+1)$]

$\geq 1 + k/2 + 1/2$ [canceling a common factor of $N(k)$ in second term]

$= 1 + (k+1)/2$

This establishes the inductive step of the proof. Thus, Harmonic series is a **divergent infinite** series.
Another Problem

Problem: Suppose that you have two different algorithms for solving a problem. To solve a problem of size $n$, the first algorithm uses exactly $n(\log n)$ operations and the second algorithm uses exactly $n^{3/2}$ operations. As $n$ grows, which algorithm uses fewer operations?

Solution:

Proof: We have already seen earlier that $n(\log n)$ is $O(n^{3/2})$, but the opposite is not true. So, for large $n$, the first algorithm uses fewer operations.

Let us see how to solve $n(\log n) < n^{3/2}$. Take logarithm on both sides to get the following: $\log_2 n + \log_2 \log_2 n < (3/2) \log_2 n$, or $\log_2 \log_2 n < 0.5 \log n$.

Note that both sides of the above inequality are monotonically increasing with $n$ and Put $n = 4$ to get $2 + 1 = (3/2) \cdot 2 = 3$.

We get that the first algorithm uses fewer operations for all $n > 4$. 
Prove that every $2^n \times 2^n$ $(n > 1)$ chessboard can be tiled with T-ominos.

A T-omino is a tile pictured as shown in the figure.
Basic Case: $2^2 \times 2^2$ Board
Basic Case: $2^2 \times 2^2$ Board
Basic Case: $2^2 \times 2^2$ Board
Basic Case: $2^2 \times 2^2$ Board
**Inductive Step**

Assume that we can tile any board of size \(2^2\) by \(2^2\) up to \(2^n\) by \(2^n\). We must show that this implies that we can tile a board of size \(2^{n+1}\) by \(2^{n+1}\).

Proof: Divide the \(2^{n+1}\) by \(2^{n+1}\) board into 4 parts, each of size \(2^n\) by \(2^n\). Since we know that each of these boards can be tiled, then we can put them together to tile the \(2^{n+1}\) by \(2^{n+1}\) board.
Show that every amount of postage that is at least 12 cents can be made from 4-cent and 5-cent stamps.

**Basis Step:** If the postage is 12 cents then we can use three 4 cent stamps. If the postage is 13 cents we can use two 4 cent stamps and one 5 cent stamp. If the postage is 14 cents we can use one 4 cent stamp and two 5 cent stamps. If the postage is 15 cents we can use three 5 cent stamps.

\[
\begin{align*}
12 &= 4+4+4 \\
13 &= 4+4+5 \\
14 &= 4+5+5 \\
15 &= 5+5+5 
\end{align*}
\]
Show that every amount of postage that is at least 12 cents can be made from 4-cent and 5-cent stamps.

**Inductive Step:**
Assume that we can represent any postage value from 12 cents up to \( k \) cents. We must show that we can represent a postage of \( k+1 \) cents.

We have already shown that we can represent any value < 16 cents so assume \( k+1 \geq 16 \) cents.

If \( k+1 \geq 16 \) then \( (k+1) - 4 \geq 12 \). So by the inductive hypothesis, we can represent \( (k+1) - 4 \) using \( m \) 4-cent stamps and \( n \) 5-cent stamps where \( m,n \in \mathbb{N} \).

\[
(k+1) - 4 = 4m + 5n \implies k+1 = 4 + 4m + 5n = 4(m+1) + 5n.
\]

So we can represent \( k+1 \) cents of postage with \( (m+1) \) 4-cent stamps and \( n \) 5-cent stamps.