Functions

Rosen (6th Edition) 2.3
Let $A$ and $B$ be nonempty sets.

- A function $f$ from $A$ to $B$ is an assignment of exactly one element of $B$ to each element of $A$. We write $f : A \to B$.

- We write $f(a) = b$ if $b$ is the unique element of $B$ assigned by the function, $f$, to the element of $a$ of $A$.

- Example: $f : \mathbb{Z} \to \mathbb{Z}$, where for each $x \in \mathbb{Z}$, $f(x) = x^2$. 
Domain, Codomain and Range

- If $f$ is a function from $A$ to $B$, we say that $A$ is the **domain** of $f$ and $B$ is the **codomain** of $f$.
- If $f(a) = b$, we say that $b$ is the **image** of $a$ and $a$ is a **pre-image** of $b$. Note that the image of $x$ is **unique**. But there can be more than one preimages for $y$.
- The **range** of $f$ is the set of all images of elements of $A$.
- **Note** that the image of $a$ is unique. But there can be more than one preimages for $b$.
- Also, if $f$ is a function from $A$ to $B$, we say that $f$ maps $A$ to $B$. 
Addition and Multiplication

Let \( f_1 \) and \( f_2 \) be functions from \( A \) to \( \mathbb{R} \) (real numbers).

\[ f_1 + f_2 \text{ is defined as } (f_1 + f_2)(x) = f_1(x) + f_2(x). \]

\[ f_1 f_2 \text{ is defined as } (f_1 f_2)(x) = f_1(x)f_2(x). \]

(Two real-valued functions with the same domain can be added and multiplied.)

**Example:** \( f_1(x) = x^2 ; f_2(x) = x+x^2 \)

\[ (f_1 + f_2)(a) = f_1(a) + f_2(a) = a^2 + a + a^2 = 2a^2 + a \]

\[ (f_1 f_2)(a) = f_1(a)f_2(a) = (a^2)(a+a^2) = a^3+a^4 \]
Are \( f_1 + f_2 \) and \( f_1 f_2 \) Commutative?

Prove: \((f_1 + f_2)(x) = (f_2 + f_1)x \) where \( x \in \mathbb{R} \)

Proof: Let \( x \in \mathbb{R} \) be an arbitrary element in the domain of \( f_1 \) and \( f_2 \). Then \((f_1 + f_2)(x) = f_1(x) + f_2(x) = f_2(x) + f_1(x) = (f_2 + f_1)(x)\).

Prove: \((f_1 f_2)(x) = (f_2 f_1)(x) \) where \( x \in \mathbb{R} \)

Proof: Let \( x \in \mathbb{R} \) be an arbitrary element in the domain of \( f_1 \) and \( f_2 \). Then \((f_1 f_2)(x) = f_1(x)f_2(x) = f_2(x)f_1(x) = (f_2 f_1)(x)\).
Let $f$ be a function from the set $A$ to the set $B$ and let $S$ be a subset of $A$.

The **image of $S$** is the subset of $B$ that consists of the images of the elements of $S$.  $f(S) = \{f(s) \mid s \in S\}$.

Example: $S = \{a_1, a_2\}$
Image of $S = \{b_1, b_2\}$
One-to-one function

A function $f$ is said to be one-to-one, or injective, if and only if $f(x) = f(y)$ implies that $x = y$ for all $x$ and $y$ in the domain of $f$.

∀$a_0, a_1 \in A$

$a_0 \neq a_1 \rightarrow f(a_0) \neq f(a_1)$
Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$, where $f(x) = 2x$

Prove that $f$ is one-to-one

Proof: We must show that $\forall ~ x_0, x_1 \in \mathbb{Z} \ (f(x_0) = f(x_1)) \rightarrow (x_0 = x_1)$.

Consider arbitrary $x_0$ and $x_1$ that satisfy $f(x_0) = f(x_1)$. By the function’s definition we know that $2x_0 = 2x_1$. Dividing both sides by 2, we get $x_0 = x_1$. Therefore $f$ is one-to-one.
Let $g: \mathbb{Z} \rightarrow \mathbb{Z}$, where $g(x) = x^2 - x - 2$

Prove that $g$ is one-to-one.

Not True! To prove a function is not one-to-one it is enough to give a counter example such that $f(x_1) = f(x_2)$ and $x_1 \neq x_2$.

Counter Example: Consider $x_1 = 2$ and $x_2 = -1$.

Then $f(2) = 2^2 - 2 - 2 = 0 = f(-1) = (-1)^2 + 1 - 2$. Since $f(2) = f(-1)$ and $2 \neq -1$, $g$ is not one-to-one.
Define \( g(a,b) = (a-b, a+b) \)

Prove that \( g \) is one-to-one.

Proof: We must show that \( g(a,b) = g(c,d) \) implies that \( a=c \) and \( b=d \) for all \((a,b)\) and \((c,d)\) in the domain of \( g \).

Assume that \( g(a,b) = g(c,d) \), then \((a-b, a+b) = (c-d, c+d)\) or

\[
\begin{align*}
    a-b &= c-d \quad \text{(eq 1)} \quad \text{and} \\
    a+b &= c+d \quad \text{(eq 2)}
\end{align*}
\]

\(a = c-d+b\) from the first equation and \(a+b = (c-d+b) + b = c+d\) using the second equation.

\[2b = 2d \implies b=d\]

Then substituting \( b \) for \( d \) in the second equation results in \( a+b = c+b \implies a=c\).
More Function Definitions and Proofs

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Onto Function

A function $f$ from $A$ to $B$ is called **onto, or surjective**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

$\forall b \in B \exists a \in A$ such that $f(a) = b$

Say, $f(x) = x^2$ from $\mathbb{N}$ to $\mathbb{N}$. Is $f$ onto?
Let \( g: \mathbb{R} \rightarrow \mathbb{R} \), where \( g(x) = 3x - 5 \)

Prove: \( g(x) \) is onto.

Proof: Let \( y \) be an arbitrary real number. For \( g \) to be onto, there must be an \( x \in \mathbb{R} \) such that \( y = g(x) = 3x - 5 \).

Solving for \( x \) in terms of \( y \), \( x = \frac{1}{3}(y + 5) \) which is a real number. Since \( x \) exists, then \( g \) is onto.
Let $f: \mathbb{R} \rightarrow \mathbb{R}$, where $f(x) = x^2 + 1$

Prove or disprove: $f$ is onto

Counter Example: Let $y = 0$, then there does not exist an $x$ such that $f(x) = x^2 + 1$ since $x^2$ is always positive.
Define $g(a, b) = (a - b, a + b)$ for real numbers

Prove that $g$ is onto.

Proof: We must show that $\forall (c, d) \exists (a, b)$ such that $g(a, b) = (c, d)$.

What we must do is show how to generate real-valued $a$ and $b$ for any values of $c$ and $d$.

Starting from the definition if $(c, d) = g(a, b) = (a - b, a + b)$ then $c = a - b$, $d = a + b$;
\[
c + d = (a - b) + (a + b) = 2a; \quad c - d = (a - b) - (a + b) = -2b
\]
Solving for $a$ and $b$ we get $a = (c + d)/2$ and $b = (d - c)/2$

Both of these values are defined for real numbers so we have shown that an ordered pair $(a, b)$ exists such that $g(a, b) = (c, d)$ for any real values of $c$ and $d.$
One-to-one Correspondence

The function $f$ is a **one-to-one correspondence** or a **bijection**, if it is both one-to-one and onto.

Bijection?
Inverse Function, $f^{-1}$

Let $f$ be a \textbf{one-to-one correspondence} from the set $A$ to the set $B$. The inverse function of $f$ is the function that assigns to an element $b$ belonging to $B$ the unique element $a$ in $A$ such that $f(a) = b$. $f^{-1}(b) = a$ when $f(a) = b$

Example:

\[ b = f(a) = 3(a-1) \]
\[ a = f^{-1}(b) = \left(\frac{b}{3}\right) + 1 \]
**Define** \( g(a, b) = (a-b, a+b) \)

Find the inverse function \( g^{-1} \)

We want \( g^{-1}(a-b, a+b) = (a,b) \)

Try

\[
g^{-1}(a-b, a+b) = \left( \frac{(a-b)+(a+b)}{2}, \frac{(a+b)-(a-b)}{2} \right)
\]

\[
g^{-1}(a-b, a+b) = \left( \frac{2a}{2}, \frac{2b}{2} \right) = (a, b)
\]
Examples

Is each of the following: a function? one-to-one? Onto? Have an inverse? on the real numbers?

\( f(x) = \frac{1}{x} \)
not a function \( f(0) \) undefined

\( f(x) = \sqrt{x} \)
not a function since not defined for \( x < 0 \)

\( f(x) = x^2 \)
is a function, not 1-to-1 (-2, 2 both go to 4), not onto since no way to get to the negative numbers, no inverse.
Composition of Functions

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Quick Review

A function from set A to set B assigns exactly one element of B to each element of A.

A function f is said to be **one-to-one, or injective**, if and only if \( f(x) = f(y) \rightarrow x = y \) for all \( x \) and \( y \) in the domain of f.

A function f from A to B is called **onto, or surjective**, if and only if \( \forall b \in B \exists a \in A \text{ such that } f(a) = b \)

The function f is a **one-to-one correspondence or a bijection**, if it is **both** one-to-one and onto.
Composition of Functions

Let $g$ be a function from the set $A$ to the set $B$ and let $f$ be a function from the set $B$ to the set $C$. The **composition** of the functions $f$ and $g$, denoted by \( f \circ g \), is defined by \( (f \circ g)(a) = f(g(a)) \).

**Example:** Let $f$ and $g$ be functions from $\mathbb{Z}$ to $\mathbb{Z}$ such that $f(x) = 2x+3$ and $g(x) = 3x+2$

\[
(f \circ g)(4) = f(g(4) = f(3(4)+2) = f(14) = 2(14)+3 = 31
\]
Examples of Composition

1. Consider \( f : \mathbb{R} \to \mathbb{R} \), where for each \( x \in \mathbb{R} \), \( f(x) = 2x + 3 \), and \( g : \mathbb{R} \to \mathbb{R} \), where for each \( x \in \mathbb{R} \), \( g(x) = 3x - 2 \). Then, \((f \circ g)(x) = f(3x - 2) = 2(3x - 2) + 3 = 6x - 1\)

2. \( f : \{0, 1, 2\} \to \{0, 1, 2\} \), where \( f(0) = 1 \), \( f(1) = 2 \), \( f(2) = 0 \); \( g : \{0, 1, 2\} \to \{1, 2, 3\} \), where \( g(0) = 1 \), \( g(1) = 2 \), \( g(2) = 3 \). Then \( g \circ f (0) = g(f(0)) = g(1) = 2 \), \( g \circ f (1) = g(f(1)) = g(2) = 3 \), and so on.
Suppose that $g: \mathcal{A} \rightarrow \mathcal{B}$ and $f: \mathcal{B} \rightarrow \mathcal{C}$ are both onto. Is $(f \circ g)$ onto?

Proof: We must show that $\forall y \in \mathcal{C}$, $\exists x \in \mathcal{A}$ such that $y = (f \circ g)x = f(g(x))$.

Let $y$ be an arbitrary element of $\mathcal{C}$. (1) Since $f$ is onto, then $\exists b \in \mathcal{B}$ such that $y = f(b)$. (2) Now, since $g$ is onto, then $b = g(x)$ for some $x \in \mathcal{A}$. Hence $y = f(b) = f(g(x)) = (f \circ g)x$ for some $x \in \mathcal{A}$. Hence, $(f \circ g)$ is onto.
Suppose that \( g: A \to B \) and \( f: B \to C \), and \( f \) and \((f \circ g)\) are onto, is \( g \) onto?

Counter Example:

Let \( A \) be the set of natural numbers \( (\mathbb{N}) \), \( B \) be the set of integers \( (\mathbb{Z}) \) and \( C \) be the set of squares of integers \( (\mathbb{Z}^2) \) where assume \( g(x) = -x \) and \( f(x) = x^2 \). Then \( g: \mathbb{N} \to \mathbb{Z} \), and \( f: \mathbb{Z} \to \mathbb{Z}^2 \), \((f \circ g): \mathbb{N} \to \mathbb{Z}^2 \). \((f \circ g)(a) = f(-a) = a^2\) is onto, \( f(b) = b^2 \) is onto, but \( g(a) = -a \) is not onto since we can only get non-positive integers.

Note: Domain and Codomain of functions play an important role.
Let $f$ be a function from set $A$ to set $B$. Let $S$ and $T$ be subsets of $A$. Show that $f(S \cap T) \subseteq f(S) \cap f(T)$.
Let $f$ be a function from set $A$ to set $B$. Let $S$ and $T$ be subsets of $A$. Show that $f(S \cap T) \subseteq f(S) \cap f(T)$.

Let $b \in f(S \cap T)$. Then $\exists a$ in $S \cap T$ such that $b = f(a)$. Since $a \in S \cap T$, then $a \in S$ and $a \in T$. Since $a \in S$, then $b \in f(S)$. Since $a \in T$, then $b \in f(T)$. Therefore $b \in f(S) \cap f(T)$.

**Question:** Is it true that $f(S) \cap f(T) \subseteq f(S \cap T)$? Try.
Suppose that \( g: A \to B \) and \( f: B \to C \) are both one-to-one. Is \( (f \circ g) \) one-to-one?

Is \( (f \circ g) = (g \circ f) \)?

Suppose that \( g: A \to B \) and \( f: B \to C \) and \( f \) and \( (f \circ g) \) are one-to-one, is \( g \) one-to-one?

Try to write arguments to prove or disprove.
Show that $(f \circ g)$ is one-to-one if $g: A \rightarrow B$ and $f: B \rightarrow C$ are both one-to-one.

Proof: We must show that, $\forall x, y \in A$, $x \neq y \rightarrow (f \circ g)(x) \neq (f \circ g)(y)$. Let $x, y$ be distinct elements of $A$. Then, since $g$ is one-to-one, $g(x) \neq g(y)$.

Now, since $g(x) \neq g(y)$ and $f$ is one-to-one, then $f(g(x)) = (f \circ g)(x) \neq f(g(y)) = (f \circ g)(y)$.

Therefore $x \neq y \rightarrow (f \circ g)(x) \neq (f \circ g)(y)$, so the composite function is one-to-one.
Is \((f \circ g) = (g \circ f)\)?

\[\text{No. A counter example is let } f: \mathbb{Z} \rightarrow \mathbb{Z} \text{ and } g: \mathbb{Z} \rightarrow \mathbb{Z} \text{ and } g(a) = a^2 \text{ and } f(a) = 2a. \text{ Then}
\]
\[(f \circ g)(3) = f(g(3)) = f(9) = 18
\]
\[(g \circ f)(3) = g(f(3)) = g(6) = 36\]
Suppose that $g: A \to B$ and $f: B \to C$ and $f$ and $(f \circ g)$ are one-to-one, is $g$ one-to-one?

Proof (by contradiction): From the assumptions $(f \circ g): A \to C$ and $\forall x, y \in A, x \neq y \to (f \circ g)(x) \neq (f \circ g)(y)$ since $(f \circ g)$ is one-to-one.

Assume that $g$ is not one-to-one. Then there must exist distinct $x, y \in A$ such that $g(x) = g(y)$.

Since $g(x) = g(y)$, then certainly $f(g(x)) = (f \circ g)(x) = f(g(y)) = (f \circ g)(y)$.

This contradicts our assumption that $(f \circ g)$ is one-to-one. Thus, our assumption that “$g$ is not one-to-one” is wrong. (Note that we did not need the fact that $f$ is one-to-one.)
Let $f$ be a function from $A$ to $B$. Let $S$ be a subset of $B$.
Show that $f^{-1}(S) = f^{-1}(S)$

What do we know?
f must be 1-to-1 and onto (why?)
Let $f$ be a function from $A$ to $B$. Let $S$ be a subset of $B$. Show that $f^\text{-1}(S) = \overline{f^\text{-1}(S)}$

Proof: We must show $f^{-1}(\overline{S}) \subseteq f^{-1}(S)$ and that $f^{-1}(S) \subseteq f^{-1}(\overline{S})$ (requirement to prove set equality).

Let $x \in f^{-1}(\overline{S})$. Then $x \in A$ and $f(x) \not\in S$. Since $f(x) \not\in S$, $x \not\in f^{-1}(S)$. Therefore $x \in f^{-1}(S)$.

Now let $x \in \overline{f^{-1}(S)}$. Then $x \not\in f^{-1}(S)$ which implies that $f(x) \not\in S$. Therefore $f(x) \in \overline{S}$ and $x \in f^{-1}(\overline{S})$.