Predicate Calculus
Propositional Functions

Propositional functions (or predicates) are propositions that contain variables.

Example: Let \( P(x) \) denote \( x > 3 \)

\( P(x) \) has no truth value until the variable \( x \) is bound by either

- assigning it a value or by
- quantifying it.
Assignment of values

Let Q(x,y) denote “x + y = 7”.

Each of the following can be determined as T or F.

Q(4,3) T
Q(3,2) F
Q(4,3) ∨ Q(3,2) T
¬[Q(4,3) ∧ Q(3,2)] T

Try to explain why [Remember, ∨ is OR, ∧ is AND];
you can use a truth table if you want to.
Quantifiers

Universe of Discourse, $U$: The domain of a variable in a propositional function.

Universal Quantification of $P(x)$ is the proposition: “$P(x)$ is true for all values of $x$ in $U$.”

Existential Quantification of $P(x)$ is the proposition: “There exists an element, $x$, in $U$ such that $P(x)$ is true.”
Universal Quantification of $P(x)$

$\forall x P(x)$

“for all $x$ $P(x)$”

“for every $x$ $P(x)$”

Defined as:

$P(x_0) \land P(x_1) \land P(x_2) \land P(x_3) \land \ldots$ for all $x_i$ in $U$

Example:
Let $P(x)$ denote $x^2 \geq x$

If $U$ is $x$ such that $0 < x$ then $\forall x P(x)$ is false.

If $U$ is $x$ such that $1 < x$ then $\forall x P(x)$ is true.
Existential Quantification of $P(x)$

Read as “there exists”

$\exists x P(x)$

“there is an $x$ such that $P(x)$”

“there is at least one $x$ such that $P(x)$”

“there exists at least one $x$ such that $P(x)$”

Defined as:

$P(x_0) \lor P(x_1) \lor P(x_2) \lor P(x_3) \lor \ldots$ for all $x_i$ in $U$

Example:

Let $P(x)$ denote $x^2 \geq x$

If $U$ is $x$ such that $0 < x < 1$ then $\exists x P(x)$ is false.

If $U$ is $x$ such that $0 < x \leq 1$ then $\exists x P(x)$ is true.
Quantifiers

∀xP(x)
• True when P(x) is true for every x.
• False if there is an x for which P(x) is false.

∃xP(x)
• True if there exists an x for which P(x) is true.
• False if P(x) is false for every x.
Precedence of Quantifiers

∀ and ∃ have higher precedence than all other logical operators (∨, ∨, ¬, ⊕ etc.). **Always**, to avoid confusion, use parentheses to be precise to say what you mean.

∀x P(x) ∨ Q(x) means (∀x P(x)) ∨ Q(x)
∀x P(x) ∨ Q(x) does not mean ∀x (P(x) ∨ Q(x))
Negation (it is not the case)

\( \neg \exists x P(x) \) is equivalent to \( \forall x \neg P(x) \), 
\( \neg \exists x P(x) \equiv \forall x \neg P(x) \)

True when \( P(x) \) is false for every \( x \)

False when there is an \( x \) for which \( P(x) \) is true.

Example: Let \( P(x) \) be the statement “\( x \) is a USC fan” and the Universe of Discourse be the students in our CPSC2070 class.

\( \neg \exists x P(x) \equiv \) It is not the case that there is a student in CPSC2070 who is a USC fan.

\( \forall x \neg P(x) \equiv \) For all students in CPSC2070 it is not the case that one of us is a USC fan.
Examples

Let \( T(a,b) \) denote the propositional function “a trusts b.” [does not say anything about if b trusts a]. Let \( U \) be the set of all people in the world.

Everybody trusts Bob.
\( \forall x T(x, \text{Bob}) \)
Could also say: \( \forall x \in U \ T(x,\text{Bob}) \)

Bob trusts somebody [There is at least one person in the world who is trusted by Bob]
\( \exists x T(\text{Bob},x) \)
Examples

Alice trusts herself.
\( T(\text{Alice}, \text{Alice}) \)

Alice trusts nobody.
\( \forall x \neg T(\text{Alice}, x) \)

Carol trusts everyone trusted by David.
\( \forall x (T(\text{David}, x) \rightarrow T(\text{Carol}, x)) \)

Bob trusts only Alice.
\( T(\text{Bob}, \text{Alice}) \land \forall x (x=\text{Alice} \lor \neg T(\text{Bob}, x)) \)
Bob trusts only Alice.

\[ T(\text{Bob, Alice}) \land \forall x (x=\text{Alice} \lor \neg T(\text{Bob, } x)) \]

Let \( p \) be “\( x=\text{Alice} \)”
q be “Bob trusts \( x \)”

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False only when Bob trusts someone who is not Alice
Quantification of Two Variables
Quantification of Two Variables
(read left to right)

\[ \forall x \forall y P(x, y) \text{ or } \forall y \forall x P(x, y) \]
- True when \( P(x, y) \) is true for every pair \( x, y \).
- False if there is a pair \( x, y \) for which \( P(x, y) \) is false.

\[ \exists x \exists y P(x, y) \text{ or } \exists y \exists x P(x, y) \]
- True if there is a pair \( x, y \) for which \( P(x, y) \) is true.
- False if \( P(x, y) \) is false for every pair \( x, y \).

\( \forall \) stands for “for all”, \( \exists \) stands for “there exists (one)”
Quantification of Two Variables

$\forall x \exists y P(x, y)$
- True when for every $x$ there is a $y$ for which $P(x, y)$ is true.
- False if there is an $x$ such that $P(x, y)$ is false for every $y$.

$\exists x \forall y P(x, y)$
- True if there is an $x$ for which $P(x, y)$ is true for every $y$.
- False if for every $x$ there is a $y$ for which $P(x, y)$ is false.
Let $P(x,y)$ be the statement $x+y = 7$

- Is $\forall x \exists y P(x,y)$ true or false?
  - For every number $x$ we can find a number $y$ such that $x + y = 7$.
  - TRUE

- Is $\exists x \forall y P(x,y)$ true or false?
  - For every number $x$ we can find a number $y$ such that $x + y \neq 7$.
  - FALSE

Note: There is an implicit assumption in these answers. Can you tell what? Hint: consider the domain of the variables $x$ and $y!$
Let $P(x, y)$ be the statement $x + y = 7$

Is $\forall x \exists y P(x, y)$ true or false?

- For every number $x$ we can find a number $y$ such that $x + y = 7$.
- TRUE [assuming $x$ and $y$ are integers in the range $[-\infty, \infty]$. But if $x, y$ are in the range $[0, \infty]$, then it is FALSE; consider $x = 20$, there is no $y$ to satisfy $x + y = 7$]

Is $\exists x \forall y P(x, y)$ true or false?

- For every number $x$ we can find a number $y$ such that $x + y \neq 7$.
- FALSE [assuming $x$ and $y$ are integers in the range $[-\infty, \infty]$, pick $x=3$, then $y = 4$ will satisfy $x+y = 7$. If $x, y$ are in the range $[-\infty, 0]$, then also it is FALSE]
Determine the truth value of each of these statements if $\mathbb{U}$ is all of the integers.

$\forall n \exists m \ (n^2 < m)$
For every integer $n$ there exists integer $m$ such that $n^2 < m.$  
TRUE

$\exists m \forall n \ (n^2 < m)$
There exists an $m$ such that for every integer $n$, $n^2 < m.$  
FALSE

$\exists n \exists m \ (n^2 + m^2 \neq 6)$
There exists an integer $n$ and an integer $m$ such that $n^2+m^2 \neq 6.$  
FALSE
Determine the truth value of each of these statements if $U$ is all of the integers.

$\exists n \forall m \ (nm = m)$

There exists at least one integer, $n$, such that for every integer $m$, $nm = m$.

**TRUE, $n = 1$.**

$\forall m \forall n \ (mn = 20)$

For every integer $m$ and every integer $n$, $mn = 20$.

**FALSE**

$\exists m \exists n \ (mn = 20)$.

There exists an integer $m$ and an integer $n$ such that $mn = 20$.

**TRUE, $4*5 = 20$, $2*10 = 20$, $20*1=20$**

OK, how about $\exists m \exists n \ (mn = -20)$. 
Determine the truth value of each of these statements if $\mathcal{U}$ is all of the integers.

$\forall m \exists n \ (nm < 0)$
For every integer $m$, we can find an integer $n$, such that $nm < 0$. **FALSE, $m = 0$ is the counter example.**

$\forall m \exists n \ (m + n = 0)$
For every integer $m$, there exists an integer $n$ such that $m + n = 0$. **TRUE**

$\exists n \forall m \ (m + n = 0)$
There exists an integer $n$, that for every integer $m$, $m + n = 0$. **FALSE, no one integer added to all other integer would always equal zero!**
Example

Let P(x), Q(x), and R(x) be the statements “x is a lion,” “x is fierce,” and “x drinks coffee,” respectively. Assuming that the domain consists of all creatures, express the statements in the argument “Some fierce creatures do not drink coffee” using quantifiers and P(x), Q(x), and R(x).

\[ \forall x(P(x) \rightarrow Q(x)). \] [All lions are fierce]
\[ \exists x(P(x) \land \neg R(x)). \] [Some lion does not drink coffee]
\[ \exists x(Q(x) \land \neg R(x)). \] [Some fierce creatures do not drink coffee]

Note: Notice that the second statement cannot be written as \[ \exists x(P(x) \rightarrow \neg R(x)). \] The reason is that P(x) \rightarrow \neg R(x) is true whenever x is not a lion, so that \[ \exists x(P(x) \rightarrow \neg R(x)) \] is true as long as there is at least one creature that is not a lion, even if every lion drinks coffee. Similarly, the third statement cannot be written as \[ \exists x(Q(x) \rightarrow \neg R(x)). \]
Practice with Quantifiers
Example 1

Let

\( P(x) \) be the statement: “\( x \) is a Clemson student”
\( Q(x) \) be the statement: “\( x \) is ignorant”
\( R(x) \) be the statement: “\( x \) wears red”

and \( U \) is the set of all people.

No Clemson students are ignorant.

\( \forall x (P(x) \rightarrow \neg Q(x)) \)

\( \forall x (\neg P(x) \lor \neg Q(x)) \)

OK by Implication equivalence.

\( \neg \exists x (P(x) \rightarrow Q(x)) \)

Does not work. Why?

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\(\neg \exists x (P(x) \rightarrow Q(x))\) Does not work. Why?

No Clemson students are ignorant \(\forall x (P(x) \rightarrow \neg Q(x))\)

Start with
\[
\neg \exists x (P(x) \rightarrow Q(x))
\]
\[
\iff \forall x \neg (P(x) \rightarrow Q(x))
\]
\[
\iff \forall x \neg (\neg P(x) \lor Q(x))
\]
\[
\iff \forall x (\neg \neg P(x) \land \neg Q(x))
\]
\[
\iff \forall x (P(x) \land \neg Q(x))
\]

Let us check what is meant by \(\forall x (P(x) \land \neg Q(x))\)
DeMorgan’s Laws for Quantifiers

\[ \neg (P \lor Q) \iff (\neg P \land \neg Q) \]
\[ \neg (P \land Q) \iff (\neg P \lor \neg Q) \]
\[ \neg \forall x P(x) \iff \exists x P(x) \]
\[ \exists x P(x) \iff \forall x \neg P(x) \]

Note: Be careful when you apply them remembering precedence of operators (better to use parentheses always to correctly represent what you want to say)
∀ \( x \) (\( P(x) \land \neg Q(x) \))

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Only true if everyone is a Clemson student and is not ignorant.
Example 1

P(x) be the statement: “x is a Clemson student”
Q(x) be the statement: “x is ignorant”
R(x) be the statement: “x wears red”
and U is the set of all people.

No Clemson students are ignorant.

∀x(P(x) → ¬Q(x)) OK
∀x(¬P(x) ∨ ¬Q(x)) Implication equivalence.
¬∃x(P(x) → Q(x)) Was not equivalent
¬∃x(P(x) ∧ Q(x)) Is equivalent. Why?
Examples 1

No Clemson students are ignorant.
\( \forall x (P(x) \rightarrow \neg Q(x)) \)  Correct

\( \neg \exists x (P(x) \land Q(x)) \)

\( \iff \forall x \neg (P(x) \land Q(x)) \)  DeMorgan for quantifiers

\( \iff \forall x (\neg P(x) \lor \neg Q(x)) \)  DeMorgan

\( \iff \forall x (P(x) \rightarrow \neg Q(x)) \)  Implication equivalence
Let
P(x) be the statement: “x is a Clemson student”
Q(x) be the statement: “x is ignorant”
R(x) be the statement: “x wears red”
and U is the set of all people.

All ignorant people wear red.
\[ \forall x (R(x) \rightarrow Q(x)) \]
Is this right? No! This says there may be some ignorant people wearing orange!
\[ \forall x (Q(x) \rightarrow R(x)) \]
Examples 3

Let
\( P(x) \) be the statement: “x is a Clemson student”
\( Q(x) \) be the statement: “x is ignorant”
\( R(x) \) be the statement: “x wears red”
and \( U \) is the set of all people.

No Clemson student wears red.
\( \forall x(P(x) \rightarrow \neg R(x)) \)
\( \forall x(R(x) \rightarrow \neg P(x)) \)
Both are correct since one is the contraposition of the other.