1. (a) (5) Define \((A - B)\) to be those elements in set \(A\) but not in set \(B\). Use set membership tables to determine what elements are contained in \((A \cap (B - A))\). Use set membership tables to determine what elements are contained in \((B \cap (A - B))\).

\[
\begin{array}{cccc}
A & B & B - A & A \cap (B - A) \\
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
\end{array}
\quad
\begin{array}{cccc}
A & B & A - B & B \cap (A - B) \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
\end{array}
\]

There are no elements in \(A \cap (B - A)\). Null set.
There are no elements in \(B \cap (A - B)\). Null set

(b) (5) What can you say about the sets \(A\) and \(B\) given the information below. You should use statements like “A \(\subseteq\) B” or “A = B” or “B \(\subset\) A” or “No conclusion” or “A \(\cap\) B = \(\phi\)” or “A = \(\phi\)” or “B = \(\phi\)”.

\[
\begin{align*}
A \cup B &= A? \\
B &= \phi \text{ or } B \subseteq A \\
A \cap B &= A? \\
A &= B \\
A - B &= B - A? \\
A &= B \\
A - B &= A? \\
A \cap B &= \phi \\
A \cap B &= B \cap A? \\
\text{No conclusion}
\end{align*}
\]

2. (5) List the values of the sets below. Let \(A = \{n^2 : n \in \mathbb{P} \land n \leq 5\} = \{1, 4, 9, 16, 25\}\) and \(B = \{n^4 : n \in \mathbb{P} \land n \leq 5\} = \{1, 16, 81, 256, 625\}\)

\[
\begin{align*}
A \cup B &= \{1, 4, 16, 25, 81, 256, 625\} \\
A \cap B &= \{1, 16\} \\
A - B &= \{4, 9, 25\} \\
B - A &= \{81, 256, 625\} \\
A \cap B &= \{4, 9, 25, 81, 256, 625\} \\
A \cup B &= \{4, 9, 25\}
\end{align*}
\]

3. (10) Consider sets \(A = \{1, 3, 7\}\), \(B = \{a, b\}\) and \(C = \{\&, ^\}\). Enumerate the members of the Cartesian product:

\[
\begin{align*}
B \times A \times C : &\{(a,1,\&), (a,1,^\&), (a,3,\&), (a,3,^\&), (a,7,\&), (a,7,^\&), \\
& (b,1,\&), (b,1,^\&), (b,3,\&), (b,3,^\&), (b,7,\&), (b,7,^\&)\}
\end{align*}
\]

\[
\begin{align*}
C \times B \times A : &\{(\&,a,1), (\&,a,3), (\&,a,7), (\&,b,1), (\&,b,3), (\&,b,7), \\
& (^\&,a,1), (^\&,a,3), (^\&,a,7), (^\&,b,1), (^\&,b,3), (^\&,b,7)\}
\end{align*}
\]

\[
\begin{align*}
A \times C \times B : &\{(1,\&,a), (1,\&), (1,^\&,a), (1,^\&,b), (3,\&,a), (3,\&), (3,^\&,a), (3,^\&,b), \\
& (7,\&,a), (7,\&), (7,^\&,a), (7,^\&,b)\}
\end{align*}
\]
4. (a) (10) Explain clearly why the function: \( f(x) = e^x \) or \( f(x) = 2^x \) or \( f(x) = 10^x \)
from the set of real numbers to the set of real numbers is not invertible, but if the codomain is restricted to the set of positive real numbers, the resulting function is invertible. Your reasons should be very clear and specific. (Hint: sketch the graph.)

\[ f(x) = 10^x : \] Since the function’s codomain is only on the positive real numbers, attempting to invert this function would result in something that is not a function from \( \mathbb{R} \rightarrow \mathbb{R} \), since the inverse would be undefined for negative input. If we limit the codomain to only positive reals, the inverted function’s domain is limited to onl positive reals and it is defined on all input within this domain. (Andrew Zhang)

\[ f(x) = 2^x : \] For a function to be invertible, it must be one-to-one and onto. For all negative real numbers (say \( x = -n \)), \( 2^x = 1/2^n \). \( 2^x \) will never be negative and therefore doesn’t map to negative \( \mathbb{R} \) in the codomain. It’s not invertible (can’t take the inverse). However, when the codomain is only positive real numbers, \( 2^x \) will yield a value for all real numbers that always maps to that codomain (positive real numbers). (Mary Grace Glenn)

(b) (5) As suggested in part (a) above, let the codomain of \( f(x) \) be restricted to the set of positive real numbers. Let \( g(y) \) be the inverse of \( f(x) \) described above. Define \( g(y) \). That is:

\[
\begin{align*}
  f(x) &= e^x & g(y) &= \ln(y) \quad \text{for domain } \mathbb{R}^+ \text{ and codomain } \mathbb{R} \\
  f(x) &= 2^x & g(y) &= \log_2(y) \quad \text{for domain } \mathbb{R}^+ \text{ and codomain } \mathbb{R} \\
  f(x) &= 10^x & g(y) &= \log_{10}(y) \quad \text{for domain } \mathbb{R}^+ \text{ and codomain } \mathbb{R}
\end{align*}
\]

5. Use mathematical induction to show that:

\[
\sum_{i=0}^{n} 2^i = 2^{n+1} - 1
\]

(5) Basis Step: Let \( n=0 \).

Left hand side: \( \sum_{i=0}^{n} 2^i = 2^0 = 1 \)  Right hand side: \( 2^{0+1} - 1 = 2 - 1 = 1 \)

Therefore, the equation is true for \( n=0 \).

(10) Inductive Step: Assume equation is true for \( n=k \). Prove true for \( n=(k+1) \).

Assume: \( \sum_{i=0}^{k} 2^i = 2^{k+1} - 1 \)  To prove: \( \sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1 = 2^{(k+2)} - 1 \)
Proof:

\[ \sum_{i=0}^{k+1} 2^i = \sum_{i=0}^{k} 2^i + 2^{k+1} = (2^{k+1} - 1) + 2^{(k+1)} \]

\[ = 2 \cdot (2^{k+1}) - 1 \]

\[ = 2^{(k+2)} - 1 \]

Therefore, the equation is true for all \( k \).

6. (15) Consider the following closed form solutions to sums:

\[ \sum_{i=0}^{n} a \cdot r^i = \frac{a \cdot r^{n+1} - a}{r - 1}, r \neq 1 \]

\[ \sum_{i=0}^{n} i = \frac{n(n+1)}{2} \]

\[ \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \]

\[ \sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4} \]

Use the formulas above to evaluate the following sums. Show clearly how you arrived at your answer. Specify which formula you used.

(a) Use equation 2 above.

\[ 36 + 40 + 44 + 48 + \ldots + 516 = 4 \left( \sum_{i=0}^{129} i - \sum_{i=0}^{8} i \right) \]

\[ = 4 \left( 129 \cdot 130 / 2 - (8 \cdot 9 / 2) \right) \]

\[ 40 + 44 + 48 + 52 + \ldots + 516 = 4 \left( \sum_{i=0}^{129} i + 1 + \ldots + 129 \right) - \left( \sum_{i=0}^{9} i \right) \]

\[ = 4 \left( 129 \cdot 130 / 2 - (9 \cdot 10 / 2) \right) \]

\[ 44 + 48 + 52 + 56 + \ldots + 516 = 4 \left( \sum_{i=0}^{129} i + 2 + \ldots + 129 \right) - \left( \sum_{i=0}^{10} i \right) \]

\[ = 4 \left( 129 \cdot 130 / 2 - (10 \cdot 11 / 2) \right) \]

(b) Use equation 1 above.

\[ 1 + 3 + 9 + 27 + 81 + 243 + \ldots + 729 = 3^0 + 3^1 + 3^2 + 3^3 + \ldots + 3^6 \Rightarrow a=1, r=3, n=6 \]

\[ = (3^7 - 1) / 2 \]

\[ 1 + 3 + 9 + 27 + 81 + 243 + \ldots + 2187 = 3^0 + 3^1 + 3^2 + 3^3 + \ldots + 3^7 \Rightarrow a=1, r=3, n=7 \]

\[ = (3^8 - 1) / 2 \]
Expand the following double summation and calculate the sum.

\[
\sum_{i=1}^{2} \sum_{j=0}^{3} (i - j) = \sum_{i=1}^{2} [i - 0 + (i - 1) + (i - 2) + (i - 3)] = \sum_{i=1}^{2} (4i - 6)
\]

\[
= 4 \sum_{i=1}^{2} i - \sum_{i=1}^{2} 6 = 4(1 + 2) - 2(6) = 12 - 12 = 0
\]

\[
\sum_{i=1}^{3} \sum_{j=0}^{2} (i - j) = \sum_{i=1}^{3} [(i - 0) + (i - 1) + (i - 2)] = \sum_{i=1}^{3} (3i - 3)
\]

\[
= 3 \sum_{i=1}^{3} i - \sum_{i=1}^{3} 3 = 3(6) - 9 = 9
\]

\[
\sum_{i=0}^{2} \sum_{j=1}^{3} (i - j) = \sum_{i=0}^{2} [(i - 1) + (i - 2) + (i - 3)] = \sum_{i=0}^{2} (3i - 6)
\]

\[
= 3 \sum_{i=0}^{2} i - \sum_{i=0}^{2} 6 = 3(1 + 2) - 3(6) = 9 - 18 = -9
\]

7. (10) Software Correctness. The goal of one of the verification conditions that arise in verifying an example piece of code is the following, where $S$ and $S'$ are some strings: $|S'| + 1 \leq \text{Max\_Depth}$; The givens are:

   I. $|S| \leq \text{Max\_Depth}$;
   II. $|S| > 0$;
   III. $S = <X> \circ S'$;

Which of the above givens are necessary and sufficient to prove the goal?

(a) I only,  (b) II only,  (c) III only,  (d) I and II only,  (e) I and III only

Answer: (a) 5 pts, (c) 5 points, (e) 10 points

8. (10) Specify the order of complexity of each of the following algorithms. Select among: $O(1), O(\log n), O(n), O(n \log n), O(n^2), O(n^3), O(2^n), O(n!), \text{"None of the above".}$

   O(n) Calculating the average of an array of n integers.
   O(1) Finding the largest value in a sorted array of n integers.
   O(\log n) Finding a specific value in a sorted array of n integers.
   O(n^2) Sorting an array of n integers using Bubble Sort.
   O(n^3) Printing the values of an (n x n) array.
   O(n) Taking the sum of the diagonal of an (n x n) array.
   O(2^n) Printing all values of an n-bit register.
   O(n!) Printing all permutations of the elements of a set with n values.
Taking the sum of the diagonal of an \((n \times n)\) array plus all values above the diagonal.

Printing the elements of one of the diagonals of an \((n \times n \times n)\) array.

9. Evaluate the expressions below. Your answer should involve one or more of the following letters: M, G, T, Z, P, E, K, Y. The first one has been done for you.

\[2^{29} = 512M\]

\[\log_2(64P) \times 32^2 = \log(2^6 \times 2^{50}) \times (2^5)^2 = \log(2^{56}) \times 2^{10} = 56K\]

\[\log_2(T/128K) = \log(2^{40}/2^{17}) = \log(2^{23}) = 23\]

\[(M / G) \times (Y / K) = (2^{20} / 2^{30}) \times (2^{80} / 2^{10}) = (1 / 2^{10}) \times (2^{70}) = 2^{60} = E\]

\[(2^{57} / 16^{10}) \times 512 = (2^{57} / (2^4)^{10}) \times 2^9 = (2^{57} / 2^{40}) \times 2^9 = 2^{26} = 64M\]

Number of ways you can answer a T/F test with 48 questions. \(= 2^{48} = 256T\)

\[2^{39} = 512G\]

\[\log_2(64Y) \times 8^4 = \log(2^6 \times 2^{80}) \times (2^3)^4 = \log(2^{86}) \times 2^{12} = 86 \times 4K = 344K\]

Number of ways you can answer a T/F test with 68 questions. \(= 2^{68} = 256E\)

\[\log_2(P/128K) = \log(2^{50}/2^{17}) = \log(2^{33}) = 33\]

\[(M / G) \times (P / G) = (2^{20} / 2^{30}) \times (2^{50} / 2^{30}) = (1 / 2^{10}) \times (2^{20}) = 2^{10} = K\]

\[(2^{77} / 32^{10}) / 512 = (2^{77} / (2^5)^{10}) / 2^9 = (2^{77} / 2^{40}) / 2^9 = 2^{37} / 2^9 = 2^{28} = 256M\]

\[2^{19} = 512K\]

\[\log_2(64P) \times 32^4 = \log(2^6 \times 2^{50}) \times (2^5)^4 = \log(2^{56}) \times 2^{20} = 56M\]

\[(2^{67} / 32^{10}) / 512 = (2^{67} / (2^5)^{10}) / 2^9 = (2^{67} / 2^{40}) / 2^9 = 2^{27} / 2^9 = 2^{18} = 256K\]

Number of ways you can answer a T/F test with 28 questions. \(= 2^{28} = 256M\)

\[\log_2(P/64M) = \log(2^{50}/2^{26}) = \log(2^{24}) = 24\]

\[(M / K) \times (Z / G) = (2^{20} / 2^{10}) \times (2^{70} / 2^{30}) = (2^{10}) \times (2^{40}) = 2^{50} = P\]