10/28/14

Basic Counting
Rosen 6th Ed., 5.1

Sum Rule
• If a first task can be done in \( n_1 \) ways and a second task can be done in \( n_2 \) ways, and if these tasks cannot be done at the same time, then there are \( n_1 + n_2 \) ways to do either task.
• If \( A \) and \( B \) are disjoint sets then \( |A \cup B| = |A| + |B| \)
• In general if \( A_1, A_2 \ldots A_n \) are disjoint sets, then \( |A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n| \)

Product Rule
• Suppose that a procedure can be broken down into two tasks. If there are \( n_1 \) ways to do the first task and \( n_2 \) ways to do the second task after the first task has been done, then there are \( n_1 n_2 \) ways to do the procedure.
• If \( A \) and \( B \) are disjoint sets then \( |A \times B| = |A| \cdot |B| \)
• In general if \( A_1, A_2 \ldots A_n \) are disjoint sets, then \( |A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \ldots \cdot |A_n| \)

Examples
• There are 18 math majors and 325 computer science majors at a college
  – How many ways are there to pick two representatives, so that one is a math major and the other is a computer science major?
  \( 18 \times 325 = 5850 \)
  – How many ways are there to pick one representative who is either a math major or a computer science major?
  \( 18 + 325 = 343 \)

Examples
• A multiple choice test contains 10 questions.
  – How many ways can a student answer the questions on the test if every question is answered?
  \( 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^{10} \)
  – How many ways can a student answer the questions on the test if the student can leave answers blank?
  \( 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^{10} \)

Password Example
Each user on a computer system has a password which is 6 to 8 characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?
• By the sum rule, if \( P \) is the total number of possible passwords and \( P_6, P_7, P_8 \) denote the number of passwords of length 6,7, and 8, respectively, then \( P = P_6 + P_7 + P_8 \)
Password Example

$P_6 = \text{number of six character passwords containing at least one digit}$

$= (\text{total number of six character passwords})$ $- (\text{number of six character passwords containing no digits}).$


$= 1,867,866,560$

Password Example

By the same reasoning

$P_7 = 36^7 - 26^7 = 70,332,353,920$

$P_8 = 36^8 - 26^8 = 2,612,282,842,880$

$P_6 + P_7 + P_8 = 2,684,483,063,360$

Just for fun: If a two GHz PC can check 200 million passwords a second, what is the longest time it would take to find the password to this system?

$(2,684,483,063,360/200,000,000)/(60*60)$ hours

Less than four hours

Principle of Inclusion-Exclusion

- When two tasks can be done at the same time we add the number of ways to do each of the two tasks, then subtract the number of ways to do both tasks.
- If $A$ and $B$ are not disjoint $|A \cup B| = |A| + |B| - |A \cap B|$
  - Don’t count objects in the intersection of two sets more than once!

How many bit strings of length eight either start with 1 or end with the two bits 00?

Add (number of bit strings that look like 1xxxxxxx) to the (number of bit strings that look like xxxxxx00) minus the (number of bit string that look like 1xxxxx00)

$1*2*2*2*2*2*2*1*1$

$- 1*2*2*2*2*1*1*1$

$= 2^7 + 2^6 - 2^5 = 2^5(4+2-1)$

$= 5*32 = 160$

How many positive integers with exactly three decimal digits (between 100 and 999 inclusively):

- Are divisible by 7?
- Are odd?
- Have the same three decimal digits?
- Are not divisible by 4?
- Are divisible by 3 or 4?
- Are not divisible by either 3 or 4?
- Are divisible by 3 but not by 4?
- Are divisible by 3 and 4?

How many positive integers with exactly three decimal digits (between 100 and 999 inclusively):

- Are divisible by 7?

Divisible by 7 means equal to 7n where $n \in \mathbb{Z}$

$7*15 = 105$

$7*142 = 994$

$142 - 15 + 1 = 128$
How many positive integers with exactly three decimal digits (between 100 and 999 inclusively):

- **Are odd?**
  9*10*5 = 450

- **Have the same three decimal digits?**
  9*1*1 = 9

- **Are not divisible by 4?**
  Divisible by 4? 100 = 4*25; 996 = 4*249 so 249 – 25 + 1 = 225
  9*10*10 – 225 = 900 – 225 = 675

How many positive integers with exactly three decimal digits (between 100 and 999 inclusively):

- **Are divisible by 3 or 4?**
  225 are divisible by 4
  3*34 = 102; 999 = 3*333, so 333-34+1 = 300 are divisible by 3.
  Some are divisible by 3 and 4
  108 = 9*12; 996 = 12*83; 83-9+1=75
  225+300-75 = 450

How many positive integers with exactly three decimal digits (between 100 and 999 inclusively):

- **Are not divisible by either 3 or 4?**
  900 – 450 = 450

- **Are divisible by 3 but not by 4?**
  300 divisible by 3
  75 divisible by 3 and 4
  200 – 75 =225

How many positive integers with exactly three decimal digits (between 100 and 999 inclusively):

- **Are divisible by 3 and 4?**
  75 are divisible by 12