Sequences & Summations

Consider any function that maps a subset of integers to some set of values

A sequence is a discrete structure used to represent an ordered list.

A sequence is a function from a subset of the set of integers (usually either the set \{0,1,2,\ldots\} or \{1,2,3,\ldots\}) to a set S.

We use the notation \(a_n\) to denote the image of the integer n. We call \(a_n\) a term of the sequence.

Notation to represent sequence is \(\{a_n\}\)

Examples

- \{1, 1/2, 1/3, 1/4, \ldots\} or the sequence \(\{a_n\}\) where \(a_n = 1/n\), \(n \in \mathbb{Z}^+\).
- \{1, 2, 4, 8, 16, \ldots\} = \(\{a_n\}\) where \(a_n = 2^n\), \(n \in \mathbb{N}\).
- \{1, 4, 9, 16, \ldots\} = \(\{a_n\}\) where \(a_n = n^2\), \(n \in \mathbb{Z}^+\).

Common Sequences

\(\{a_n\}\) where \(a_n = \)

<table>
<thead>
<tr>
<th>Arithmetic</th>
<th>(a, a+d, a+2d, a+3d, a+4d, \ldots)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n^2)</td>
<td>1, 4, 9, 16, 25, \ldots</td>
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<tr>
<td>(n^3)</td>
<td>1, 8, 27, 64, 125, \ldots</td>
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<tr>
<td>(n^4)</td>
<td>1, 16, 81, 256, 625, \ldots</td>
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<tr>
<td>(2^n)</td>
<td>2, 4, 8, 16, 32, \ldots</td>
</tr>
<tr>
<td>(3^n)</td>
<td>3, 9, 27, 81, 243, \ldots</td>
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<tr>
<td>(n!)</td>
<td>1, 2, 6, 24, 120, \ldots</td>
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</tbody>
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Summations

- Notation for describing the sum of the terms \(a_m, a_{m+1}, \ldots, a_n\) from the sequence, \(\{a_n\}\)
  \[ a_m + a_{m+1} + \ldots + a_n = \sum_{j=m}^{n} a_j \]

  - \(j\) is the index of summation (dummy variable)
  - The index of summation runs through all integers from its lower limit, \(m\), to its upper limit, \(n\).
Examples
\[ \sum_{j=1}^{5} j = 1 + 2 + 3 + 4 + 5 = 15 \]
\[ \sum_{j=0}^{4} (j + 1) = 1 + 2 + 3 + 4 + 5 = 15 \]

Examples
\[ \sum_{j=1}^{5} j = \frac{1+1/2+1/3+1/4+1/5}{5} \]
\[ 1 + \sum_{j=2}^{5} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \]

Summations follow all the rules of addition!
\[ c \sum_{j=1}^{n} j = \sum_{j=1}^{n} cj = c(1+2+\ldots+n) = c + 2c + \ldots + nc \]

Telescoping Sums
\[ \sum_{j=1}^{n} a_j - a_{j-1} = (a_1 - a_0) + (a_2 - a_1) + \ldots + (a_n - a_{n-1}) = a_n - a_0 \]

Example
\[ \sum_{k=1}^{4} [k^2 - (k-1)^2] = (1^2 - 0^2) + (2^2 - 1^2) + (3^2 - 2^2) + (4^2 - 3^2) = 4^2 - 0 = 16 \]

Closed Form Solutions
A simple formula that can be used to calculate a sum without doing all the additions.

Example:
\[ \sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6} \]

Proof: First we note that \( k^2 - (k-1)^2 = k^2 - (k^2-2k+1) = 2k-1 \). Since \( k^2-(k-1)^2 = 2k-1 \), then we can sum each side from \( k=1 \) to \( k=n \)
\[ \sum_{k=1}^{n} [k^2 - (k-1)^2] = \sum_{k=1}^{n} (2k-1) \]
Proof (cont.)
\[
\sum_{k=1}^{n}(k^2 - (k-1)^2) = \sum_{k=1}^{n}(2k - 1)
\]
\[
\sum_{k=1}^{n}(k^2 - (k-1)^2) = \sum_{k=1}^{n}2k + \sum_{k=1}^{n}(-1)
\]
\[
n^2 - 0^2 = 2\sum_{k=1}^{n}(k) + n
\]
\[
n^2 + n = 2\sum_{k=1}^{n}(k)
\]
\[
\sum_{k=1}^{n} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}
\]

Closed Form Solutions to Sums
\[
\sum_{j=0}^{n} j = 0 + 1 + \ldots + n = \frac{n(n+1)}{2}
\]
\[
\sum_{j=0}^{n} j^2 = 0^2 + 1^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}
\]
\[
\sum_{k=0}^{n} = \frac{n^2(n+1)}{4}
\]
\[
\sum_{a} = \frac{ar^{n+1} - a}{r-1}, r \neq 1
\]

Double Summations
\[
\sum_{i,j} i = \sum_{i} i = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}
\]
\[
6 + 12 + 18 + 24 = 60
\]

Cardinality
- Earlier we defined cardinality of a set as the number of elements in the set. We can extend this idea to infinite sets.
- The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B.
- A set that is either finite or has the same cardinality as the set of natural numbers is called countable. A set that is not countable is called uncountable.

What is \[
\sum_{k=0}^{n} 2^{-k}
\]
\[
\sum_{a} = \frac{ar^{n+1} - a}{r-1}, r \neq 1
\]
Let a = 1 and r = 1/2
\[
\sum_{i} = \frac{1 - (\frac{1}{2})^{n+1}}{1 - \frac{1}{2}} = 2^{n+1} - 2
\]
\[
1 - 2^{-n} = \frac{2^{n+1}}{2^n} - \frac{2^i}{2^i}
\]
\[
= 2^{n+1} - 1
\]
What if $k$ starts at 1?

$$\sum_{j=2}^{n} 2^{-j} = \sum_{j=1}^{n} 2^{-j} - 2^{-0} = \frac{2^{n+1} - 1}{2^{n+1}} = \frac{2^{n+1} - 1}{2^{n+1}}$$

Two more closed form solutions

$$\sum_{j=2}^{n} 2^{-j} = \frac{2^{n+1} - 1}{2^{n+1}}$$

Solve $2^x - c_0 + 2^x c \sum_{j=1}^{2} 2^x$

$$2^x c_0 + 2^x c \sum_{j=1}^{2} 2^x = \frac{2^x - 1}{2^x} = \frac{2^x - 1}{2^x}$$

Find: $\sum_{k=100}^{200} k$

$$\sum_{k=100}^{200} k = \sum_{k=1}^{200} k - \sum_{k=1}^{99} k = \frac{(200)(201)}{2} - \frac{99(100)}{2} = 20100 - 49500 = 15150$$

Derive the closed form for $\sum_{k=1}^{n} k^2$

First we note that $k^3 - (k-1)^3 = 3k^2 - 3k + 1$, then we sum both sides of the equation from 1 to $n$

$$\sum_{k=1}^{n} [k^3 - (k-1)^3] = \sum_{k=1}^{n} (3k^2 - 3k + 1)$$

On the left hand side we have a telescoping sum so it is equal to $n^3$

$$n^3 = \sum_{k=1}^{n} 3k^2 - \sum_{k=1}^{n} 3k + \sum_{k=1}^{n} 1$$

$$= 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + n$$

$$= 3\sum_{k=1}^{n} k^2 - 3\left(n\left(n + 1\right)\right) + n$$

Rearranging the terms we get

$$n^3 = \sum_{k=1}^{n} 3k^2 - 3\left(n\left(n + 1\right)\right) + n$$
\[
\sum_{n=1}^{n} n^2 = \frac{1}{3} \left( n^3 + \frac{3n(n + 1)}{2} - n \right)
= \frac{1}{3} \left( 2n^2 + 3n(n + 1) - 2n \right)
= \frac{n}{3} \left( 2n^2 + 3n + 3 - 2 \right)
= \frac{n}{3} \left( 2n^2 + 3n + 1 \right)
= \frac{n(n + 1)(2n + 1)}{6}
\]