Functions

Rosen (6th Edition) 2.3

Definition of Function

Let A and B be sets.

• A function f from A to B is an assignment of exactly one element of B to each element of A.
• We write f(a) = b if b is the unique element of B assigned by the function, f, to the element of A.
• If f is a function from A to B, we write f : A → B.

Terminology

• If f is a function from A to B, we say that A is the domain of f and B is the codomain of f.
• If f(a) = b, we say that b is the image of a and a is a pre-image of b.
• The range of f is the set of all images of elements of A.
• Also, if f is a function from A to B, we say that f maps A to B.

Addition and Multiplication

• Let f₁ and f₂ be functions from A to R (real numbers).
• f₁ + f₂ is defined as (f₁ + f₂)(x) = f₁(x) + f₂(x).
• f₁ f₂ is defined as (f₁ f₂)(x) = f₁(x) f₂(x).
• (Two real valued functions with the same domain can be added and multiplied.)
• Example: f₁(x) = x²; f₂(x) = x + x²
  • (f₁ + f₂)(a) = a² + a + a² = 2a² + a
  • (f₁ f₂)(a) = (a²)(a + a²) = a³ + a⁴

Are f₁ + f₂ and f₁ f₂ Commutative?

Prove: (f₁ + f₂)(x) = (f₁ f₂)(x) where x ∈ R
Proof: Let x ∈ R be an arbitrary element in the domain of f₁ and f₂. Then (f₁ + f₂)(x) = f₁(x) + f₂(x) = f₁(x) + f₂(x) = (f₁ f₂)(x).

Prove: (f₁ f₂)(x) = (f₂ f₁)(x) where x ∈ R
Proof: Let x ∈ R be an arbitrary element in the domain of f₁ and f₂. Then (f₁ f₂)(x) = f₁(x) f₂(x) = f₂(x) f₁(x) = (f₂ f₁)(x).

Image

Let f be a function from the set A to the set B and let S be a subset of A.
The image of S is the subset of B that consists of the images of the elements of S. f(S) = {f(s) | s ∈ S}.

Example: S = {a₁, a₂}
Image of S = {b₁, b₂}
One-to-one function

A function f is said to be one-to-one, or injective, if and only if f(x) = f(y) implies that x=y for all x and y in the domain of f.

∀a₀,a₁ ∈ A
a₀ ≠ a₁ → f(a₀) ≠ f(a₁)

Let f:Z→Z, where f(x) = 2x

Prove that f is one-to-one

Proof: We must show that ∀ x₀, x₁ ∈ Z f(x₀) = f(x₁) → x₀ = x₁.

Consider arbitrary x₀ and x₁ that satisfy f(x₀) = f(x₁).
By the function’s definition we know that 2x₀ = 2x₁. Dividing both sides by 2, we get x₀ = x₁.
Therefore f is one-to-one.

Define g(a,b) = (a-b, a+b)

Prove that g is one-to-one.

Proof: We must show that g(a,b) = g(c,d) implies that a=c and b=d for all (a,b) and (c,d) in the domain of g.

Assume that g(a,b) = g(c,d), then (a-b, a+b) = (c-d, c+d) or
a-b=c-d (eq 1) and a+b = c+d (eq 2)
a = c-d+b from the first equation and
a+b = (c-d+b) + b = c+d using the second equation
2b = 2d ⇒ b=d
Then substituting b for d in the second equation results in a+b = c+b ⇒ a=c

Onto Function

A function f from A to B is called onto, or surjective, if and only if for every element b∈B there is an element a∈A with f(a) = b.

∀b∈B ∃a∈A such that f(a) = b

Let f:R→R, where f(x) = x²+1

Prove or disprove: f is onto

Counter Example: Let y = 0, then there does not exist an x such that f(x) = x² + 1 since x² is always positive.

Let g:Z→Z, where g(x) = x²-x-2

Prove that g is one-to-one.

Not True! To prove a function is not one-to-one it is enough to give a counter example such that f(x₁) = f(x₂) and x₁ ≠ x₂.

Counter Example: Consider x₁ = 2 and x₂ = -1.
Then f(2) = 2²-2-2 = 0 ≠ f(-1) = -1² + 1 -2. Since f(2) = f(-1) and 2 ≠ -1, g is not one-to-one.
Let \( g: \mathbb{R} \to \mathbb{R} \), where \( g(x) = 3x - 5 \)

**Prove: \( g(x) \) is onto.**

**Proof:** Let \( y \) be an arbitrary real number. For \( g \) to be onto, there must be an \( x \in \mathbb{R} \) such that \( y = 3x - 5 \). Solving for \( x \), \( x = \frac{y + 5}{3} \) which is a real number. Since \( x \) exists, then \( g \) is onto.

Define \( g(a,b) = (a-b, a+b) \)

**Prove that \( g \) is onto.**

**Proof:** We must show that \( \forall (c,d) \exists (a,b) \) such that \( g(a,b) = (c,d) \).

Define \( a = \frac{c+d}{2} \) and \( b = \frac{d-c}{2} \), then

\[
\begin{align*}
  c &= c + \frac{d}{2} - \frac{d}{2} = (c/2 + d/2) - (d/2 - c/2) = (c+d)/2 - (d-c)/2 = a-b \\
  d &= d + c/2 - c/2 = (d/2 + c/2) + (d/2 - c/2) = (d+c)/2 + (d-c)/2 = a+b.
\end{align*}
\]

Therefore \( g \) is onto.

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**One-to-one Correspondence**

The function \( f \) is a **one-to-one correspondence** or a **bijection**, if it is both one-to-one and onto.

**Inverse Function, \( f^{-1} \)**

Let \( f \) be a **one-to-one correspondence** from the set \( A \) to the set \( B \). The inverse function of \( f \) is the function that assigns to an element \( b \) belonging to \( B \) the unique element \( a \) in \( A \) such that \( f(a) = b \). \( f^{-1}(b) = a \) when \( f(a) = b \)

Example:

\[
\begin{align*}
  f(x) &= 3(x-1) \\
  f^{-1}(y) &= (y/3)+1
\end{align*}
\]

**Examples**

Is each of the following: a function? one-to-one? Onto? Invertible? on the real numbers?

\( f(x) = \frac{1}{x} \)
- not a function \( f(0) \) undefined

\( f(x) = \sqrt{x} \)
- not a function since not defined for \( x < 0 \)

\( f(x) = x^2 \)
- is a function, not 1-to-1 (\(-2, 2 \) both go to 4), not onto since no way to get to the negative numbers, not invertible
**Composition of Functions**

Let $g$ be a function from the set $A$ to the set $B$ and let $f$ be a function from the set $B$ to the set $C$. The **composition** of the functions $f$ and $g$, denoted by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.

**Example:** Let $f$ and $g$ be functions from $\mathbb{Z}$ to $\mathbb{Z}$ such that $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

$f \circ g(4) = f(g(4)) = f(3(4) + 2) = f(14) = 2(14) + 3 = 31$

**Quick Review**

- A function from set $A$ to set $B$ assigns exactly one element of $B$ to each element of $A$.
- A function $f$ is said to be **one-to-one, or injective**, if and only if $f(x) = f(y) \rightarrow x = y$ for all $x$ and $y$ in the domain of $f$.
- A function $f$ from $A$ to $B$ is called **onto, or surjective**, if and only if $\forall b \in B \exists a \in A$ such that $f(a) = b$.
- The function $f$ is a **one-to-one correspondence or a bijection**, if it is both one-to-one and onto.

**Suppose that $g: A \rightarrow B$ and $f: B \rightarrow C$ are both onto. Is $(f \circ g)$ onto?**

**Proof:** We must show that $\forall y \in C$, $\exists x \in A$ such that $y = f(g(x))$.

Let $y$ be an arbitrary element of $C$. Since $f$ is onto, then $\exists b \in B$ such that $y = f(b)$.

Now, since $g$ is onto, then $b = g(x)$ for some $x \in A$.

Hence $y = f(b) = f(g(x)) = (f \circ g)(x)$ for some $x \in A$.

Hence, $(f \circ g)$ is onto.

**Counter Example:** Let $A$ be the set of natural numbers, $B$ be the set of integers and $C$ be the set of squares of integers where $g(a) = -a$ and $f(b) = b^2$. Then $g: \mathbb{N} \rightarrow \mathbb{Z}$ and $f: \mathbb{Z} \rightarrow \mathbb{Z}^2$. $(f \circ g)(a) = f(-a) = a^2$ is onto, $f(b) = b^2$ is onto, but $g(a) = -1$ is not since we can only get non-positive integers.

Let $f$ be a function from set $A$ to set $B$. Let $S$ and $T$ be subsets of $A$. Show that $f(S \cap T) \subseteq f(S) \cap f(T)$.
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There exists $b \in f(S \cap T)$.

Then there exists $a$ in $S \cap T$ such that $b = f(a)$.

Since $a \in S \cap T$, then $a \in S$ and $a \in T$.

Since $a \in S$, then $b \in f(S)$.

Since $a \in T$, then $b \in f(T)$.

Therefore $b \in f(S) \cap f(T)$.

Other interesting questions

- Suppose that $g:A \to B$ and $f:B \to C$ are both one-to-one. Is $(f \circ g)$ one-to-one?
- Does $(f \circ g) = (g \circ f)$?
- Suppose that $g:A \to B$ and $f:B \to C$ and $f$ and $(f \circ g)$ are one-to-one, is $g$ one-to-one?

Show that $(f \circ g)$ is one-to-one if $g:A \to B$ and $f:B \to C$ are both one-to-one.

**Proof:** We must show that, $\forall \ x, y \in A$, $x \neq y \rightarrow (f \circ g)(x) \neq (f \circ g)(y)$.

Let $x, y$ be distinct elements of $A$. Then, since $g$ is one-to-one, $g(x) \neq g(y)$.

Now, since $g(x) \neq g(y)$ and $f$ is one-to-one, then $f(g(x)) = (f \circ g)(x) \neq f(g(y)) = (f \circ g)(y)$.

Therefore $x \neq y \rightarrow (f \circ g)(x) \neq (f \circ g)(y)$, so the composite function is one-to-one.

Inverse Image

- Let $f$ be a function from set $A$ to set $B$. Let $S$ be a subset of $B$. We define the inverse image of $S$ to be the subset of $A$ containing all pre-images of all elements of $S$.
- $f^{-1}(S) = \{a \in A \mid f(a) \in S\}$
Let $f$ be a function from $A$ to $B$. Let $S$ be a subset of $B$. Show that $f^{-1}(S) = f^{-1}(S)$

What do we know?
- $f$ must be 1-to-1 and onto

Proof:
We must show that $f^{-1}(S) \subseteq f^{-1}(S)$ and that $f^{-1}(S) \subseteq f^{-1}(S)$.

Let $x \in f^{-1}(S)$. Then $x \in A$ and $f(x) \notin S$. Since $f(x) \notin S$, $x \notin f^{-1}(S)$. Therefore $x \in f^{-1}(S)$.

Now let $x \in f^{-1}(S)$. Then $x \notin f^{-1}(S)$ which implies that $f(x) \notin S$. Therefore $f(x) \in S$ and $x \in f^{-1}(S)$