More Set Definitions and Proofs

2.1, 2.2 Rosen 6th Edition

Mental Warm up with Symmetric Difference!

- Prove \((A \oplus B) \oplus B = A\)
- \(A \oplus B\) ↔ elements in A or B but not in both.

Prove \((A \oplus B) \oplus B = A\)

First prove by using a membership table.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A \oplus B</th>
<th>(A \oplus B) \oplus B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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A and \((A \oplus B) \oplus B\) have identical membership tables.

Now we will show that \(A \subseteq (A \oplus B) \oplus B\).

Let \(e \in A\).

Either \(e\) is also \(\in B\) or \(e \notin B\).

If \(e \in B\), then \(e \notin (A \oplus B)\) so \(e\) is an element of \((A \oplus B) \oplus B\).

If \(e \notin B\), \(e\) is an element of \((A \oplus B)\) and \(e\) must be an element of \((A \oplus B) \oplus B\).

Generalized Unions

The union of a collection of sets \(A_1 \cup A_2 \cup \ldots \cup A_n\) can be written more compactly using the notation:

\[
A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i
\]

Example: Let \(A_1 = \{a, b\}; A_2 = \{a, b, c\}; A_3 = \{b, c, d\}\)

\[
\bigcup_{i=1}^{3} A_i = \{a, b, c, d\}
\]
Generalized Intersections

The intersection of a collection of sets $A_1 \cap A_2 \cap \ldots \cap A_n$ can be written more compactly using the notation:

$$ A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i $$

**Example:** Let $A_1 = \{a, b\}$; $A_2 = \{a, b, c\}$; $A_3 = \{b, c, d\}$

$$ \bigcap_{i=1}^{3} A_i = \{b\} $$

More general example.

Let $A_i = \{1, 2, 3, \ldots, i\}$

Find $\bigcup_{i=1}^{n} A_i = \{1, 2, 3, \ldots, n\}$

Find $\bigcap_{i=1}^{n} A_i = \{1\}$

Sets are collections of objects that are unordered. Order can be important so we need a different structure to represent ordered collections of objects.

Ordered n-tuples

**Ordered n-tuple**

The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has $a_1$ as its first element, $a_2$ as its second element . . . And $a_n$ as its nth element.

2-tuples are called ordered pairs.

$(10, 3) \neq (3, 10)$

Cartesian Product of A and B

Let A and B be sets. The **Cartesian Product** of A and B, denoted $A \times B$ is the set of ordered pairs $(a, b)$ where $a \in A$ and $b \in B$.

Hence $A \times B = \{(a, b) | a \in A \land b \in B\}$

**Example:** $A = \{1, 2\}$, $B = \{a, b, c\}$

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

Note that $A \times B \neq B \times A$
Cartesian Product of the sets $A_1, A_2, \ldots, A_n$

The Cartesian Product of the sets $A_1, A_2, \ldots, A_n$ denoted by $A_1 \times A_2 \times \cdots \times A_n$ is the set of ordered $n$-tuples $(a_1, a_2, \ldots, a_n)$ where $a_i$ belongs to $A_i$ for $i = 1, 2, \ldots, n$.

\[ A_1 \times A_2 \times \cdots \times A_n = \{a_1, a_2, \ldots, a_n \mid a_i \in A_i \text{ for } i=1,2,\ldots,n \} \]

Example: $A = \{0,1\}; B = \{1,2\}; C = \{0,1,2\}$

$A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$