Proofs Using Logical Equivalences
Rosen (6th Ed.) 1.2

Note: These are all Direct Proofs

Prove: \((p \land \neg q) \lor q \iff p \lor q\)

\((p \land \neg q) \lor q\) Left-Hand Statement
\(\iff q \lor (p \land \neg q)\) Commutative
\(\iff (q \lor p) \land (q \lor \neg q)\) Distributive
\(\iff (q \lor p) \land T\) Negation
\(\iff q \lor p\) Identity
\(\iff p \lor q\) Commutative

Begin with exactly the left-hand side statement
End with exactly what is on the right
Justify EVERY step with a logical equivalence

Why did we need this step?
Our logical equivalence specified that \(\lor\) is distributive on the right. This does not guarantee distribution on the left!
Ex.: Matrix multiplication
(Note that whether or not \(\lor\) is distributive on the left is not the point here.)

Prove: \(p \rightarrow q \iff \neg q \rightarrow \neg p\)

\(p \rightarrow q\) Implication Equivalence
\(\iff \neg p \lor q\) Commutative
\(\iff q \lor \neg p\) Distributive
\(\iff \neg q \lor \neg p\) Negation
\(\iff \neg q \rightarrow \neg p\) Implication Equivalence

Contrapositive

Prove: \(p \rightarrow p \lor q\) is a tautology

Must show that the statement is true for any value of \(p\) and \(q\)

\(p \rightarrow p \lor q\)
\(\iff \neg p \lor (p \lor q)\) Implication Equivalence
\(\iff (\neg p \lor p) \lor q\) Associative
\(\iff T \lor q\) Commutative
\(\iff q \lor T\) Negation
\(\iff q \lor T\) Commutative
\(\iff T\) Domination

This tautology is called the addition rule of inference.

Why do I have to justify everything?

• Note that your operation must have the same order of operands as the rule you quote unless you have already proven (and cite the proof) that order is not important.
  • \(3+4 = 4+3\)
  • \(3/4 \neq 4/3\)
  \(A*B \neq B*A\) for everything!
Prove: \((p \land q) \rightarrow p\) is a tautology

\[(p \land q) \rightarrow p\]
\[\iff \neg (p \land q) \lor p\]
\[\iff (\neg p \lor \neg q) \lor p\]
\[\iff (\neg q \lor (p \lor q)) \lor p\]
\[\iff (p \lor q) \lor \neg (p \lor q)\]
\[\iff (p \lor q) \lor \neg q\]
\[\iff T\]

Implication Equivalence
DeMorgan's
Commutative
Associative
Negation
Domination

Prove or Disprove

\[p \rightarrow q \iff p \land \neg q ???\]

To prove that something is not true it is enough to provide one counter-example.

(Something that is true must be true in every case.)

\[
\begin{array}{ccc}
p & q & p \rightarrow q & p \land \neg q \\
F & T & T & F \\
\end{array}
\]

The statements are not logically equivalent

Class Exercise: Without using truth tables, prove that \(((p \lor q) \land \neg p) \rightarrow q\) is a tautology.

\[
\begin{array}{c}
p \lor \top \iff p \\
p \lor \bot \iff p \\
p \lor p \iff p \\
\neg (p \lor q) \iff (p \land \neg q) \\
(p \land q) \iff (p \lor q) \\
\neg (p \land q) \iff (\neg p \lor \neg q) \\
\neg (p \lor q) \iff (\neg p \land \neg q) \\
\top \iff p \\
\bot \iff p \\
\top \iff T \\
\bot \iff F \\
\top \iff (\neg p \lor q) \\
\bot \iff (p \lor \neg q) \\
\end{array}
\]

Implication Equivalence
Idempotent Laws
Double Negation Law
Commutative Laws
Associative Laws
Double Negation Law
Implication Equivalence
Negation Laws
Absorption Laws
Identity Laws
Domination Laws
Idempotent Laws
DeMorgan's Laws
Absorption Laws
Identity Laws
Domination Laws
Double Negation Law
Commutative Laws
Absorption Laws
Identity Laws

Class Exercise: Without using truth tables, prove that \(((p \lor q) \land \neg p) \rightarrow q\) is a tautology.

\[(p \lor q) \land \neg p \rightarrow q\]
\[\iff \neg ((p \lor q) \land \neg p) \lor q\]
\[\iff \neg (p \lor q) \lor \neg \neg p \lor q\]
\[\iff \neg (p \lor q) \lor \neg \neg p \lor q\]
\[\iff \neg (p \lor q) \lor \neg \neg p \lor q\]
\[\iff \neg (p \lor q) \lor \neg \neg p \lor q\]
\[\iff \neg (p \lor q) \lor \neg \neg p \lor q\]
\[\iff \neg (p \lor q) \lor \neg \neg p \lor q\]
\[\iff \neg (p \lor q) \lor \neg \neg p \lor q\]

Implication Equivalence
Idempotent Laws
Double Negation Law
Commutative Laws
Associative Laws
Double Negation Law
Implication Equivalence
Negation Laws
Absorption Laws
Identity Laws
Domination Laws
Double Negation Law
Commutative Laws
Absorption Laws
Identity Laws

Normal or Canonical Forms

Rosen (6th Ed.) 1.2 (exercises)
Logical Operators

- \( \lor \) - Disjunction
- \( \land \) - Conjunction
- \( \neg \) - Negation
- \( \rightarrow \) - Implication
- \( \oplus \) - Exclusive or
- \( \iff \) - Biconditional

Do we need all these?

Functionally Complete

- A set of logical operators is called functionally complete if every compound proposition is logically equivalent to a compound proposition involving only these logical operators.

\( \land, \lor, \text{ and } \neg \) form a functionally complete set of operators.

Are \( \neg(p \lor (\neg p \land q)) \) and \( (\neg p \land \neg q) \) equivalent?

Functionally Complete

- Even though both are expressed with only \( \land, \lor, \text{ and } \neg \), it is still hard to tell without doing a proof.

- What we need is a unique representation of a compound proposition that uses \( \land, \lor, \text{ and } \neg \).

- This unique representation is called the Disjunctive Normal Form.

Are \( \neg(p \lor (\neg p \land q)) \) and \( (\neg p \land \neg q) \) equivalent?

Disjunctive Normal Form

- A disjunction of conjunctions where every variable or its negation is represented once in each conjunction (a minterm) - each minterm appears only once

Example: DNF of \( p \oplus q \) is \( (p \land \neg q) \lor (\neg p \land q) \)

Truth Table

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \oplus q )</th>
<th>( (p \land \neg q) \lor (\neg p \land q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
**Method to construct DNF**
- Construct a truth table for the proposition.
- Use the rows of the truth table where the proposition is True to construct minterms
  - If a variable is false, use the negation of the variable in the minterm
  - If the variable is true, use the propositional variable in the minterm
- Connect the minterms with \( \lor \)'s.

**How to find the DNF of \((p \land q) \rightarrow \lnot r\)**

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>(~r)</th>
<th>((p \land q) \rightarrow \lnot r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

There are five sets of input that make the statement true. Therefore there are five minterms.

**Can we show that just \( \lnot \) and \( \land \) form a set of functionally complete operands?**

Use De Morgan’s Laws on the DNF.

Example:
\[
(p \lor q) \rightarrow \lnot r
\]

\[
\iff (p \land q \land \lnot r) \lor (p \land \lnot q \land \lnot r)
\]

\[
\lor (\lnot p \land q \land \lnot r) \lor (\lnot p \land \lnot q \land \lnot r)
\]

\[
\lor (\lnot p \land \lnot q \land \lnot r)
\]

\[
\lor (\lnot p \land q \land \lnot r) \land (\lnot p \land \lnot q \land \lnot r)
\]

\[
\iff (\lnot \lnot(p \land q) \land \lnot (\lnot p \land q)) \land (\lnot \lnot (\lnot p \land q) \land \lnot (\lnot p \land q))
\]

\[
\iff (\lnot(p \land q) \land \lnot(p \land q)) \land (\lnot(p \land q) \land \lnot(p \land q))
\]

**Find an expression equivalent to \( p \rightarrow q \) that uses only conjunctions and negations.**

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>( p \rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

How many minterms in the DNF?

The DNF of \( p \rightarrow q \) is \((p \land q) \lor (\lnot p \land q) \lor (\lnot p \land \lnot q)\).

Then, applying De Morgan’s Law, we get that this is equivalent to

\[
\lnot(\lnot(p \land q) \land \lnot(p \land q) \land \lnot(p \land \lnot q)).
\]

**Now can we write an equivalent statement to \( p \rightarrow q \) that uses only disjunctions and negations?**

\[
p \rightarrow q
\]

\[
\iff \lnot(\lnot(p \land q) \land \lnot(p \land q) \land \lnot(p \land \lnot q))
\]

From Before

\[
\iff \lnot(\lnot(p \land q) \land \lnot(p \land q) \land \lnot(p \land \lnot q))
\]

DeMorgan

\[
\iff \lnot(\lnot(p \land q) \land \lnot(p \land q) \land \lnot(p \land \lnot q))
\]

Doub. Neg.

\[
\iff \lnot(\lnot(p \land q) \land \lnot(p \land q) \land \lnot(p \land \lnot q))
\]

DeMorgan