CPSC 4040/6040
Computer Graphics
Images

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Agenda

• Q05

• PA07 Questions?
Refresher from Lec22
Jean Baptiste Joseph Fourier (1768-1830)

Fourier’s Idea (1807):

“**Any** univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.”
A Sum of Sines

Our building block:
- \( f(x) = A \sin(\omega x + \phi) \)

Add enough of them to get any signal you want!

How many degrees of freedom?

What does each control?

Which one encodes the coarse vs. fine structure of the signal?

\[ f(\text{target}) = f_1 + f_2 + f_3 + \ldots + f_n + \ldots \]
Sinusoids (one-D) as Images

\[ f(x) = A \times \sin\left(\frac{2\pi ux}{N} + \phi\right) \]

Breakdown frequency by:
- \( N \) is the width of the image
- \( u \) controls number of cycles

Varying Amplitude

Varying Phase

Varying Frequency

(a) \( u = 2, \phi = 0, a = 1 \)
(b) \( u = 2, \phi = 0, a = .5 \)
(c) \( u = 2, \phi = 0, a = 0 \)

(d) \( u = 2, \phi = 0, a = 1 \)
(e) \( u = 2, \phi = \pi/4, a = 1 \)
(f) \( u = 2, \phi = \pi/2, a = 1 \)

(g) \( u = 2, \phi = 0, a = 1 \)
(h) \( u = 3, \phi = 0, a = 1 \)
(i) \( u = 4, \phi = 0, a = 1 \)
Sampling Theorem

A signal can be reconstructed from its samples, iff the original signal has no content >= 1/2 the sampling frequency - Shannon

**Aliasing** will occur if the signal is under-sampled
Examples of Frequency Spectra

(a) $f(x)$ and $|F(u)|$

(b) $f(x)$ and $|F(u)|$

(c) $f(x)$ and $|F(u)|$
DCT Basis Functions

Basis Functions

Imaged
Warmup and Questions
Why does the Gaussian give a nice smooth image, but the box filter give edgy artifacts?
Why does a lower resolution image still make sense to us? What do we lose?
What Other Signals We “Regularly” Decompose?
What Other Signals We “Regularly” Decompose?

• Example: Music
  • We think of music in terms of frequencies at different magnitudes
  • Different notes/pitches played at different volumes
Why/How does the technique of hybrid images work?

Hybrid images

Aude Oliva
Antonio Torralba
Philippe G. Schyns

http://cvcl.mit.edu/hybridimage.htm
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SIGGRAPH 2006

http://cvcl.mit.edu/hybridimage.htm
The Fourier Domain
Definition of the Fourier Domain

- Expresses an image as the sum of weighted sinusoids
  - Wavelengths are determined by image dimensions
  - Amplitudes are determined by sample values
- Fourier coefficients are complex values rather than real values
- Given a one-dimensional sequence $f$ of $N$ samples, the one-dimensional discrete Fourier transform (DFT) is given as

$$
\mathcal{F}(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \left[ \cos \left( \frac{2\pi ux}{N} \right) - j \sin \left( \frac{2\pi ux}{N} \right) \right]
$$

- $N$ is the length of a row and hence $u$ is in $[0, N-1]$
Complex Numbers

• The symbol \( j \) denotes the imaginary unit \( = \sqrt{-1} \)
  
  • \( j \) satisfies the relation \( j^2 = -1 \). Some people use \( "i" \)

• Euler’s formula gives a more compact formula:

\[
\cos(x) - j \sin(x) = e^{-jx},
\]

(9.11)

• Thus we can write:

\[
\mathcal{F}(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) \left[ \cos \left( \frac{2\pi ux}{N} \right) - j \sin \left( \frac{2\pi ux}{N} \right) \right]
\]

\[
\mathcal{F}(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi ux/N}.
\]
Forward and Inverse Are Symmetric

\[ F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x)e^{-j2\pi ux/N} \]

\[ f(x) = \frac{1}{N} \sum_{u=0}^{N-1} F(u)e^{j2\pi ux/N} \]
2D Discrete Fourier Transform

\[
\mathcal{F}(u, v) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux+vy)/N}
\]

\[
f(x, y) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \mathcal{F}(u, v) e^{j2\pi(ux+vy)/N}
\]
Interpreting the Fourier Coefficients

• Each DFT coefficient is a complex value
  • There is a single DFT coefficient for each spatial sample
  • A complex value is expressed by two real values in either Cartesian or polar coordinate space.

• Cartesian: \( R(u,v) \) is the real and \( I(u, v) \) the imaginary component

• Polar: \( A = |F(u,v)| \) is the amplitude and \( \phi(u,v) \) the phase

\[
\mathcal{F}(u, v) = R(u, v) + jI(u, v)
\]
\[
\mathcal{F}(u, v) = |F(u, v)|e^{j\phi(u,v)}
\]
Amplitude and Phase Representation of Coefficients

- Representing the DFT coefficients as magnitude and phase is a more useful for processing and reasoning.
  - The amplitude $A$ is a measure of strength or length
  - The phase $\phi$ is a direction and lies in $[-\pi, +\pi]$  
- The magnitude and phase are easily obtained from the real and imaginary values
  
\[
|\mathcal{F}(u, v)| = \sqrt{R^2(u, v) + I^2(u, v)}
\]
\[
\phi(u, v) = \tan^{-1} \left( \frac{I(u, v)}{R(u, v)} \right).
\]
Each Coefficient Corresponds to a Basis “Sinusoid”
Fourier Spectra
Example Spectra

Input Image

Frequency Image

http://cns-alumni.bu.edu/~slehar/fourier/fourier.html
Composition of Signals

Input Image

Frequency Image

http://cns-alumni.bu.edu/~slehar/fourier/fourier.html
Spectrum of Man-made Scene

- Amplitudes are generally referred to as the “spectrum” but this should be understood as the amplitude spectrum.
Amplitude and Phase Spectrum

- Typically has an extremely large dynamic range and it is typical to log-compress those values for display (as above!)

- For presentation, $F(0,0)$, is placed at the center. Low frequency components are shown near the center and frequency increases with distance from center.

Figure 9.7. DFT Spectrum.
Amplitude vs. Phase

- The amplitude spectrum contains information about the shape of objects. A strong edge in the source will generate a strong edge in the amplitude spectrum (rotated 90 degrees).

- The phase spectrum contains information about their actual location in the source.

  - Example: an image of lots of ‘Q’s will have the same amplitude spectra but not the same phase spectra.

Figure 9.9. Illustration of DFT properties.
Each Piece of the Spectrum Contributes to the Image

(a) Reconstructed from phase information only.  (b) Reconstruction from amplitude information only.

Figure 9.8. Comparison of the contribution of the amplitude and phase spectrum.
Computing the Fourier Transform

Hi, Dr. Elizabeth? Yeah, uh... I accidentally took the Fourier transform of my cat...

Meow!
The Fourier Transform Has a Continuous Interpretation

Continuous:

Fourier Transform: \( F(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} \, dx \)

Inverse Fourier Transform: \( f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega x} \, d\omega \)

Discrete (DFT):

\[ \mathcal{F}(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi ux/N} \]

(there are options in case you have trouble remembering this)
DFT Example

- Given a row profile, compute the Fourier coefficients

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>20</td>
<td>12</td>
<td>18</td>
<td>56</td>
<td>83</td>
<td>10</td>
<td>104</td>
<td>114</td>
</tr>
</tbody>
</table>

\[ \mathcal{F}(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi ux/N} \]
DFT Code

\[ F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi ux/N} \]

```java
public enum DFT_TYPE {FORWARD, INVERSE};

public Complex[][] discreteFourierTransform(Complex[][] source,
                                           DFT_TYPE type) {

    int width = source[0].length;
    int height = source.length;
    int factor = type == DFT_TYPE.FORWARD ? -1 : 1;
    Complex[][] result = new Complex[height][width];

    for (int u=0; u<width; u++) {
        for (int v=0; v<height; v++) {
            Complex coefficient = new Complex();
            for (int x=0; x<width; x++) {
                for (int y=0; y<height; y++) {
                    Complex basis = Complex.e(factor*2*Math.PI*(u*x/width + v*y/height));
                    coefficient = coefficient.plus(source[y][x].times(basis));
                }
            }
            result[v][u] = coefficient;
        }
    }

    return result;
}
```

Listing 9.5. Brute force DFT.
FFTs

- The Fast Fourier Transform (FFT) is an efficient technique to compute the DFT by clever elimination of redundant computations.

- The FFT is used in a plethora of domains to solve problems in spectral analysis, audio signal processing, computational chemistry, error correcting codes, and of course image processing.

- While various FFT techniques have been discovered, the term FFT typically refers to the Cooley-Tukey algorithm.

- This technique was discovered and rediscovered by a variety of mathematicians but was popularized by J. W. Cooley and J. W. Tukey in their 1965 publication entitled An Algorithm for the Machine Calculation of Complex Fourier Series.
Properties of the Fourier Transform
Translation, Rotation, Distributivity

- **Translation** of the source image will cause the phase spectrum to change but leave the amplitude spectrum unchanged since the phase spectrum encodes location information while the amplitude spectrum encodes shape information.

- **Rotation** of the source image corresponds to an identical rotation of the amplitude and phase spectra.

- **Distributivity.** The Fourier transform is distributive over addition (not multiplication):

\[
\mathcal{F}(f + g) = \mathcal{F}(f) + \mathcal{F}(g)
\]
Linearity

\[ \mathcal{F}[ax(t) + by(t)] = a\mathcal{F}[x(t)] + b\mathcal{F}[y(t)] \]

- Fourier transform of a real signal is symmetric about the origin
- The energy of the signal is the same as the energy of its Fourier transform

See Szeliski 3.4
Translation, Rotation, Distributivity

Figure 9.10. Properties of the DFT under translation, rotation, and linear combination.
Periodicity

- The DFT is periodic

  - Sinusoids have infinite, repeating extent and so the DFT ‘image’ is infinite and repeated (tiled)

\[ F(u, v) = F(u + M, v) = F(u, v + N) = F(u + M, v + N), \]  \hspace{1cm} (9.16)

- Implications:

  - Since the source image wraps around itself (why?), the source image will appear to have edges where no edges actually exist.

  - For example, the left side of an image is bright and the right side is dark, there will be a false edge created when the image is tiled:

    - Cause: Bright left side is placed adjacent to dark right side.

  - These artificial edges will then be reflected in the amplitude spectrum and give a false characterization of the power spectrum of the source image.
Example of Periodicity

(a) Tiled source.

(b) Amplitude spectrum of the source.

Figure 9.11. Periodicity effects of the DFT.
Image Processing in the Frequency Domain
Frequency Domain Filtering

- Images can be processed in either the spatial or frequency domain
  - Frequency domain filters can achieve the same results as spatial filters by altering the DFT coefficients directly
  - Frequency domain filtering can be generalized as the multiplication of the spectrum $F$ of an image by a transfer function $H$.

$$G(u, v) = F(u, v) \cdot H(u, v),$$

- Most frequency domain filters are zero-phase. This means that the phase spectrum is not changed; only the amplitude spectrum is changed
First Change the Spectrum, then Reconstruct
Low and High Pass Filtering
The Further from the Center of the Spectrum, the Higher Frequencies
The Convolution Theorem

• Convolution in the spatial domain corresponds to **multiplication** in the Fourier domain.

\[ f \otimes h \Leftrightarrow \mathcal{F} \cdot H \]

• This has strong computational implications

• Recall that convolution of an NxN image with MxM kernel is \( O(M^2N^2) \) in the spatial domain.

• What is the performance in the Fourier domain? \( O(N^2)!! \)
Convolution Theorem Example

\[ f \otimes h \]

\[ F \otimes H \]
Recall: Why does the Gaussian give a nice smooth image, but the box filter give edgy artifacts?
Filtering with Gaussian
Filtering with Box
Ideal Low Pass Filtering

\[ \mathcal{H}(u, v) = \begin{cases} 
1 & \sqrt{u^2 + v^2} \leq r_c, \\
0 & \text{otherwise},
\end{cases} \quad (9.23) \]

- Here, \( r_c \) is the cutoff frequency.
- The term **ideal** does not mean that this filter is optimal for low pass filtering.
- An ideal low pass filter is ideal in the sense that it has an exact cutoff above which all terms are exactly zero and below which all terms are set to unity.

![Diagram](image)

Figure 9.15. Ideal low-pass filter in (a) the frequency domain and (b) the spatial domain.
Ideal Low Pass Filtering Example

(a) Filtered magnitude spectrum.  
(b) Reconstructed spatial domain image.

Figure 9.14. Ideal low-pass filtering in the frequency domain.
Fourier Transform Pairs

\[ F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi sx} \, dx \]
Lec24 Required Reading
• Hunt, Ch.10
• House, 5.3