CP SC 8810
Data Visualization

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Lecture 25
Topological Features

Nov. 25, 2014

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Agenda

• Lab05 Questions?

• Project Presentations: I will distribute instructions / formats soon.
Refresher from Lec24
Definition

Illustration of a symmetric second-order tensor as linear operator. The tensor is uniquely determined by its action on all unit vectors, represented by the circle in the left image. The eigenvector directions are highlighted as black arrows. In this example one eigenvalue ($l_2$) is negative. As a consequence all vectors are mirrored at the axis spanned by eigenvector $e_1$. The eigenvectors are the directions with strongest normal deformation but no directional change.
Comparison: Ellipsoids vs. superquadrics (Kindlmann)

Color map \( \begin{pmatrix} R \\ G \\ B \end{pmatrix} = c_l \begin{bmatrix} |e_x^1| \\ |e_y^1| \\ |e_z^1| \end{bmatrix} + (1 - c_l) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \)

This is half of the brain, looking at the posterior part of the corpus callosum, which is the main bridge between the two hemispheres. And with the superquadrics, you can see that on the surface of the corpus callosum, the glyphs have more of a planar component, but on the inside, they're basically very linear.
Geometry-Based Tensor Field Vis
Hyperstreamlines [Delmarcelle, Hesselink 1992/93]

- Streamlines defined by eigenvectors
- Direction of streamline by major eigenvector
- Visualization of the vector field defined by major eigenvector
- Other eigenvectors define cross-section
Hyperstreamlines

• Let $\mathbf{T}(\mathbf{x})$ be a (2nd order) symmetric tensor field
  – real eigenvalues, orthogonal eigenvectors

• Hyperstreamline: by integrating along one of the eigenvectors

• **Important: Eigenvector fields are not vector fields!**
  – eigenvectors have no magnitude and no orientation (are bidirectional)
  – the choice of the eigenvector can be made consistently as long as eigenvalues are all different
  – Hyperstreamlines can intersect only at points where two or more eigenvalues are equal, so-called **degenerate points.**
Red – major
Green – minor
Hyperstreamlines

Widely used in diffusion tensor imaging tractography

Hyperstreamlines rendered as tubes with elliptic cross section, radii proportional to 2^{nd} and 3^{rd} eigenvalue

[Shen and Pang 2004]
• Idea of hyperstreamlines:
  – Major eigenvector describes direction of diffusion with highest probability density
Problem of Hyperstreamlines

• Ambiguity in (nearly) isotropic regions:
  – Partial voluming effect, especially in low resolution images (MR images)
  – Noise in data
  – Solution: tensorlines

[Weinstein, Kindlmann 1999]

Tensorline
Hyperstreamline
Arrows: major eigenvector

• Advection vector
• Stabilization of propagation by considering
  • Input velocity vector
  • Output velocity vector (after application of tensor operation)
  • Vector along major eigenvector
• Weighting of three components depends on anisotropy at specific position:
  • Linear anisotropy: only along major eigenvector
  • Other cases: input or output vector
• **Tensorlines** [Weinstein, Kindlmann 1999]
  
  – **Advection vector**
  
  – **Stabilization of propagation by considering**
    - Input velocity vector
    - Output velocity vector (after application of tensor operation)
    - Vector along major eigenvector
  
  – **Weighting of three components depends on anisotropy at specific position:**
    - Linear anisotropy: only along major eigenvector
    - Other cases: input or output vector
• Tensorlines

Tensorlines: Yellow   Hyperstreamlines: Cyan
Hyperstreamlines

[Prckovska et al. 2010]

Hybrid visualization: hyperstreamlines + glyphs

Good for some non-symmetric tensor visualization where the rotational components can be encoded by the glyphs
Exploration of the Brain’s White Matter Pathways with Dynamic Queries

David Akers, Anthony Sherbondy, Rachel Mackenzie, Robert Dougherty, Brian Wandell

Stanford University
Exploration of the Brain's White Matter Pathways with Dynamic Queries

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Tensor orientation (0 degrees). The TEND dispersion becomes much greater for highly anisotropic tensors as the orientation becomes more oblique. As expected, the dispersion is relatively independent of orientation for more isotropic tensors.

**Human brain white matter tractography**

**Commissural pathways**

The majority of interhemispheric pathways are in the corpus callosum (CC), which has the highest anisotropy and moderate curvature. A comparison of corpus callosum tract estimates that were generated using TEND, STT, and a combination of TEND and STT defined by Equation (4) is shown in Figure 3 for Subjects 1 and 3. The apparent tracts were generated by seeding a region situated at the midline of the corpus callosum, and covering few adjacent sagittal slices. A FA map was used to separate corpus callosum from the surrounding tissue. Generally, tracking using STT or deflection resulted in fibers curving and running up toward the cingulate cortex (Fig. 3a). These results are similar to those obtained by Basser et al. [2000]. By increasing the weighting of the incoming vector in Eq (9), the estimated pathways emerge from the region of intersection with corona radiata at a lower position and reveal connections between more lateral regions of the two hemispheres (Fig. 3b and c).

**Projection pathways**

The connections between the cerebral cortex and the spine, including the corticospinal tracts, are a part of Figure 3.

Fibers of corpus callosum (Subject 1, top row; Subject 3, bottom row). The pathways were seeded on the midline of the corpus callosum in the sagittal plane. The fiber trajectories were terminated if they reached regions with FA lower than 0.15 or if the angle between two consecutive steps was larger than 45 degrees. The voxels intersected by estimated fiber trajectories were labeled and the resulting volume was rendered. Different tract reconstructions were obtained using a: STT, b: TEND, c: tensor-lines (Equation (4) w/ \( f = 0, g = 0.3 \)). c: illustrates that a combination of deflection and incoming vector resulted in fibers that connect more lateral regions of the two hemispheres.
Topological Features
What is Topology?

- Field of mathematics which studies properties which are **preserved under continuous transformations**.
  - Stretching, bending = continuous changes.
  - Tearing, gluing = discontinuous changes.
- Also called: “Rubber sheet” geometry.
- Studies the connectedness of a space.
What are Features in Field Data?

- Spatial Segmentation / Classification
- “Interesting” positions, neighborhoods, curves, surfaces, etc.
- How can this help visualization?
  - Compute features, and then augment a visualize with them to aid the user.
Why Spatial Segmentation

• Classifying spatial domain based on the scalar values is always desired.
  – Reduce the information overloaded
  – Identify unique features and properties

• Classification in DVR
  – Transfer function design

Are there any other region segmentation method?
There are Other Features

• Other than “same material has similar property and different materials have different properties”, there are some features that can be defined by the scalar values within a small neighborhood of certain spatial position

• Consider a height field, what places do people care about?
Scalar Field Topology
1D Case

• Let us get back to the simple 1D case
1D Case

- Let us find out the local minimum/maximum

Zero derivatives
1D Case

• They partition the domain into monotonic regions
How About 2D Case?

Pre-image of an iso-value: Iso-contours
We Want to Extract Similar Information

Q: Which iso-contours are interesting?
Q: Summarize the evolution of iso-contours?
Topology

• These local minimum and maximum are called “critical points” of the scalar functions.

• Their connection forms the topology of the scalar field, which provides a partition scheme of the spatial domain.

• Each segment has the equivalent homogeneous behavior, e.g. monotonic for 1D case.

• This is similar for 2D and 3D scalar fields
Scalar Field Analysis

• Here is a more formal definition
• Given a scalar field $f$
  – Gradient vector
    $$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}$$
  – When not zero
    – Points in the direction of quickest ascend
    – Always perpendicular to the iso-contours (or level sets) of $f$
• If $\nabla f (p) = 0$,
  – $p$ is a critical point
  – $f(p)$ is a critical value
Scalar Field Analysis

• A critical point $p$ is isolated if there exists a neighborhood of $p$ such that $p$ is the only critical point in the neighborhood.

• Classification of fundamental critical points in 2D:
  - Local minima
  - Saddle
  - Local maxima
Detection of Critical Points

3D saddles can have two distinct configurations

1-saddle  2-saddle
Scalar Field Analysis

• A function is a **Morse function** if it is smooth and all of its critical points are isolated and non-degenerate
  – Typically a good assumption for scientific data
  – A non-Morse function can be made Morse by adding small but random noise
Key development in topological data analysis (TDA)

1. Abstraction of the data: topological structures and their combinatorial representations

2. Separate features from noise: persistent homology

Two Types of Topological Structures

- Reeb Graph/Contour Tree/Merge Tree
- Morse-Smale Complex

2D Scalar function
Level-Set Topology
Reeb Graphs, Contour Trees, and Merge Trees
Example – dunking a doughnut

- $f(p) = z$ (height function)

Shape analysis is a special case of scalar field analysis.
Example – dunking a doughnut
Example – dunking a doughnut
Example – dunking a doughnut
Example – dunking a doughnut
Example – dunking a doughnut
How Does it Work?
How Does it Work?

Level sets obtaining by sweeping along Z direction
How Does it Work?

Critical Points

Maximum
Saddle
Saddle
Minimum
Reeb Graph
Reeb Graph

- Vertices of the graph are critical points
- Arcs of the graph are connected components (cylinders in domain) of the level sets of $f$, contracted to points

- Two-step algorithm
  - Locate critical points
  - Connect critical points
Figure 1: (Top row) Simplified Reeb graphs of the Dancer, Malaysian Goddess, Happy Buddha; and David together with two close-ups showing a tiny tunnel at the base of David’s leg. The pseudo-colored surfaces show the function used for computing the Reeb graph. The transparent models show the structure of the Reeb graph and its embedding. (Bottom row) The Heptoroid model and two levels of resolution for the Reeb graph of the Asian Dragon model.
Reeb graphs and genus

• The number of loops in the Reeb graph is equal to the surface genus
• To count the loops, simplify the graph by contracting degree-1 vertices and removing degree-2 vertices
Some More Reeb Graph Examples
Gradient-Field
Topology
Morse and Morse-Smale Complexes
Morse-Smale Complex-2D
Morse-Smale Complex-2D
Morse-Smale Complex-2D
Morse-Smale Complex-2D

Descending Manifold
Morse-Smale Complex-2D

Cell of the Morse-Smale complex
Morse-Smale Complex-2D

Decomposition into monotonic regions
Combinatorial Structure 2D

- Nodes of the MS complex are exactly the critical points of the Morse function
- Saddles have exactly four arcs incident on them
- All regions are quads
  - Boundary of a region alternates between saddle-extremum
  - 2k minima and maxima

3D MS Complex cell
Applications

Molecular surface segmentation
Applications

Rayleigh-Taylor turbulence analysis
Morse-Smale Complexes

Figure: Topology simplification applied on electron density data for a hydrogen atom: the input has a large number of critical points, several of which are identified as being insignificant and removed by repeated application of two atomic operations. Features are identified by the surviving critical points and enhanced in a volume rendered image by an automatically designed transfer function.

Gyulassy, Natarajan, Pascucci, Bremer, Hamann, 2005
Figure 11: (Upper-left) Puget Sound data after topological noise removal. (Upper-right) Data at persistence of 1.2% of the maximum height. (Lower-left) Data at persistence 20% of the maximum height. (Lower-right) View-dependent refinement (purple: view frustum).
Fig. 5. A single timestep of a dataset of a simulated Raleigh-Taylor instability simulating the mixing of two fluids. This timestep has a resolution of $1152 \times 1152 \times 1000$ and is an early timestep of the simulation. The data is noisy, therefore we perform a 5% persistence simplification to remove “excess features.” We compute the complex for the entire dataset, and the inset shows a small subsection of the data with selected nodes and arcs of the complex. Minima and maxima (blue and red spheres) and their saddle connections trace out the bubble structure in the data. The maxima represent isolated pockets of high-density fluid that have crossed the boundary between the two fluids. The structural complexity is overwhelming, but our prototype allows interactive exploration and visualization, and selective inclusion/omission of user-specified components of the MS complex.
Vector Field Topology
Topological Skeleton and Morse Decomposition
Motivation

• Abstract representation of flow field
• Characterization of global flow structures
• Basic idea (steady case):
  • *Interpret flow in terms of streamlines*
  • *Classify them w.r.t. their limit sets*
  • *Determine regions of homogenous behavior*
• Graph depiction
• Fast computation (not always)
Steady Vector Fields
Critical Points

Figure 1: Classification criteria for critical points. $R_1$ and $R_2$ denote the real parts of the eigenvalues of the Jacobian, $I_1$ and $I_2$ the imaginary parts.

Image: Surface representations of 2- and 3-dimensional fluid flow topology, Helman & Hesselink
Sectors & Separatrices

Figure 11: Example of sector type identification

Source: A topology simplification method for 2D vector fields. Xavier Tricoche, Gerik Scheuermann, & Hans Hagen

\[ I = 1 + \frac{e - h}{2} \]
Topological Skeleton

- Graph connecting the critical points using separatrices
Poincaré Index

- **Poincarè index** \( I(\Gamma, V) \) of a simple closed curve \( \Gamma \) in the plane relative to a continuous vector field is the number of the positive field rotations while traveling along \( \Gamma \) in positive direction.

- By continuity, always an integer
- The index of a closed curve around multiple fixed points will be the sum of the indices of the fixed points

[Tricoche Thesis 2002]
Limit Sets

- Limit sets reveal the long-term behaviors of vector fields, correspond to flow recurrence

- The **limit sets** are:

  \[ \alpha(x) = \bigcap_{t<0} \text{cl}(\varphi((-\infty, t), x)) \]

  point (or curve) reached after **forward** integration by streamline seeded at \( x \)

  \[ \omega(x) = \bigcap_{t>0} \text{cl}(\varphi((t, \infty), x)) \]

  point (or curve) reached after **backward** integration by streamline seeded at \( x \)
Repellor and Attractor Manifolds
Stable Manifolds of Ocean Data
Stable Manifolds of Ocean Data

1 time, depth slice:
3600x2400 mesh
30-40 mins / 3GB mem
~25k critical points
~39k cycles
Vector Field Topology

- Idea:
  Do not draw “all” streamlines, but only the “important” streamlines

- Show only topological skeletons

- Important points in the vector field: critical points

- Critical points:
  - Points where the vector field vanishes: \( v = 0 \)
  - Points where the vector magnitude goes to zero and the vector direction is undefined
  - Sources, sinks, …

- The critical points are connected to divide the flow into regions with similar properties

- Structure of particle behavior for \( t \to \infty \)
Vector Field Topology

1. Find critical points
   • pretty much iso-value algorithm
   • but with a twist - since three components are zero
   • find iso-values for each component and then only consider cells where all three intersect
   • not enough - sub-divide potential cells until a certain bound is reached.
Vector Field Topology

2. classify critical points

• according to what is happening in the neighborhood - attracting or repelling or a combination thereof

• determined by derivative of velocity

• if positive then things move away

• if negative things come closer

• this is 1D
Vector Field Topology

- Example of a topological graph of 2D flow field
Vector Field Topology

- Further examples of topology-guided streamline positioning
Vector Field Data Compression

Source: [Theisel et al. Eurographics 2003]
Periodic Orbits

• Curve-type (1D) limit set
• Attracting / repelling behavior

**Poincaré map:**
- Defined over cross section
- Map each position to next intersection with cross section along flow
- Discrete map
- Cycle intersects at fixed point
- Hyperbolic / non-hyperbolic
Periodic Orbits in the Ocean
Periodic Orbits in the Ocean

Data Credit: Mathew Maltrud from the Climate, Ocean and Sea Ice Modelling program at Los Alamos National Laboratory (LANL) and the BER Office of Science UV-CDAT team
3D Flow Topology

- Fixed points

- Can be characterized using 3D Poincaré index

- Both line and surface separatrices exist
Saddle Connectors

• Multiple separating surfaces may lead to occlusion problems

• Idea: reduce visual clutter by replacing stream surfaces with streamlines of interest

• Saddle Connector:
  – Separating surfaces intersection integrated from two saddle points of opposite indices (inflow vs. outflow surface)
  – Intersection is a streamline

Source: Theisel et al. Vis 03
Vector field topology

• 3D topology

Saddle connectors:
Tensor Field Topology
Topological Skeleton in 2D

We only consider the topology for $2^{nd}$ symmetric tensor fields!

*Image by Xavier Tricoche*
Degenerate Points

• The topology for 2\textsuperscript{nd} symmetric tensor fields is extracted by identifying their degenerate points and their connectivity.

• A point \( p \) is a degenerate point of the tensor field \( T \) iff the two eigenvalues of \( T(p) \) are equal to each other.
  – There are infinite many eigenvectors at \( p \).
  – Hyperstreamlines cross each other at degenerate points
Degenerate Points in 2D

Three linear types exist

- Trisector
- Wedge I
- Wedge II

None of these patterns would be possible in vector fields!

Discontinuity of local orientation
Degenerate Points in 2D

- Find degenerate points
- Form \( \tilde{D} := D - \frac{1}{2} \text{trace}(D)I_2 \Rightarrow \tilde{D} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \)
- Solve in each cell for \( \tilde{D}(x, y) = 0 \)

\[
D = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}
\]

\[
D - \frac{1}{2} \text{trace}(D)I_2 = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} - \begin{pmatrix} \frac{T_{11} + T_{22}}{2} & 0 \\ 0 & \frac{T_{11} + T_{22}}{2} \end{pmatrix} = \ldots
\]
Degenerate Points in 2D

• Classifying tensor degenerate points

\[
\begin{pmatrix}
\alpha(x, y) & \beta(x, y) \\
\beta(x, y) & -\alpha(x, y)
\end{pmatrix}
= \begin{pmatrix}
\alpha_1 x + \alpha_2 y & \beta_1 x + \beta_2 y \\
\beta_1 x + \beta_2 y & -(\alpha_1 x + \alpha_2 y)
\end{pmatrix}
\]

– Depending on the determinant of \[ \begin{pmatrix}
\alpha_1 & \alpha_2 \\
\beta_1 & \beta_2
\end{pmatrix} \]
  • >0 wedge
  • <0 trisector
  • =0 higher-order degenerate points
A few degenerate points used in tensor field design

\[
\begin{pmatrix}
\alpha_1 x + \alpha_2 y & \beta_1 x + \beta_2 y \\
\beta_1 x + \beta_2 y & -(\alpha_1 x + \alpha_2 y)
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\alpha_1 & \alpha_2 \\
\beta_1 & \beta_2
\end{pmatrix}
\]

>0 wedge
<0 trisector
=0 higher-order degenerate points
[Delmarcelle and Hesselink, 1994]
Degenerate Points in 2D

- Tensor index: wedges
Degenerate Points in 2D

- Tensor index: trisectors
Degenerate Points in 2D

- Tensor index: trisectors
Separatrices

Hyperbolic sectors $n_h$
Parabolic sectors $n_p$

Index $I = 1 - \frac{n_h}{2}$
Separatrices

- Find degenerate points

- Form \( \tilde{\mathbf{D}} := \mathbf{D} - \frac{1}{2} \text{trace}(\mathbf{D}) \mathbf{I}_2 \Rightarrow \tilde{\mathbf{D}} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \)

- Solve in each cell for \( \tilde{\mathbf{D}}(x, y) = 0 \)

- Compute separatrices
  - Linear analysis at each singularity \( \begin{cases} \alpha(x, y) \approx \alpha_1 x + \alpha_2 y \\ \beta(x, y) \approx \beta_1 x + \beta_2 y \end{cases} \)
  - Determine angular coordinate of separatrices
    \( \beta_2 u^3 + (\beta_1 + 2\alpha_2) u^2 + (2\alpha_1 - \beta_2) u - \beta_1 = 0 \)
    \( u := \tan \theta \) (Delmarcelle and Hesselink, 1994)
  - Integrate separatrices (standard ODE solver with embedded orientation consistency check)
Separatrices

\[
\begin{align*}
\sin 2\theta & = \beta_1 \cos \theta + \beta_2 \sin \theta \\
\cos 2\theta & = \alpha_1 \cos \theta + \alpha_2 \sin \theta
\end{align*}
\]
Lec26
Required Reading
• TBA!