Probability

- Random Experiment
  - Outcome varies in an unpredictable fashion when the experiment is repeated under the same conditions
  - Requires an experimental procedure and a set of observations
  - Has an outcome

- Sample space, $S$, is the set of all possible outcomes

- An event, $A$, consists of any subset of outcomes in the sample space

- We want to assign probabilities of specific outcomes or events

- Probability theory gives us mathematical tools to develop and use probability models

- The sample space $S$ might be discrete or continuous

Discrete Exp. - Toss a coin 3 times & record the # Heads
$S = \{HHH, HHT, HT, THH, H, T\}$

Cont. Exp. - Obtain a random # in the range $[0,1]$
Sequential Experiments: Support 10 numbers

Selected at random from the interval \([0,1]\).

Find the prob. That the first 5 numbers are less than \(\frac{1}{4}\), and the last 5 numbers are greater than \(\frac{1}{2}\).

Assuming independent events

\[ P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap P(A_6 \cap \ldots \cap A_{10}) \]...

The experiment is to pull a uniform random number.

We are interested in specific events

\[ A_k = \{ X_k < \frac{1}{4} \} \quad \text{in} \ 1 = k \ldots 5 \]

\[ = \{ x_k \geq \frac{1}{2} \} \quad \text{in} \ 6 = k \ldots 10 \]

\[ P[A_k] \geq P[A_{k+1}] \geq \ldots \geq P[A_{10}] \geq \frac{1}{2} \]

Sequential Exp -

\[ P[A_1 \cap \ldots \cap A_{10}] \leq \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \]
Sequential exp.

**Bernoulli Trial** - perform an experiment once and note if an event occurs or does not occur.

**Binomial prob. law assigns probability of**

\[ \text{Binomial Trials} \]

\[ P(x) = \binom{n}{k} p^k (1-p)^{n-k} \]

\[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \]

**Exp.** - Toss a coin 3 times and record the # heads.

**What is the prob. of 2 heads?**

\[ P_3(2) = \binom{3}{2} p^2 (1-p) \quad \text{where} \quad p = \frac{1}{2} \]

\[ = \frac{3!}{2!1!} \left( \frac{1}{2} \right)^2 \left( \frac{1}{2} \right) = 3 \left( \frac{1}{2} \right)^3 = \frac{3}{8} \]
**Probability examples**

Over a communications channel, the BER is $10^{-3}$. The transmitter sends each bit 3 times (i.e., error correction codes are used). The receiver takes a majority vote to decide on what was actually sent. So the scheme is able to correct for a single error but not for 2 errors. Find the probability that the receiver will make an incorrect decision.

Model a Transmission as a Bernoulli Trial. A success $\Rightarrow$ Error we want The probability of 2 successes in 3 trials

$$P[k \geq 2] = \binom{3}{2} (0.001)^2 (0.999) + \binom{3}{3} (0.001)^3 \approx 3 \times 10^{-6}$$

**Binomial Probability Law:** Prob of $k$ successes in $n$ independent Bernoulli Trials:

$$P_n(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

where $\binom{n}{k}$ is the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k! (n-k)!}$$
TCP retransmits when loss occurs. Assume that TCP will attempt to transmit a segment a total of 10 times before giving up. If the packet loss rate is 2%, what is the probability that a packet will not be successfully transmitted?

Using the geometric probability law:

Define a success as a successful transmission:

\[ P = 0.98 \quad 1 - P = 0.02 \]

\[ P(\text{m trials before a success}) = (1 - p)^{k-1} P \]

\[ P(\text{m} = k) = (1 - p)^{k-1} P \]

Probability that \( k \) or more trials required:

\[ P(\text{m} > k) = \sum_{m=k+1}^{\infty} (1 - p)^{m-1} P \]

Remember:

\[ \sum_{k=0}^{\infty} q^k = \frac{1}{1-q} \]

\[ = \frac{p \cdot q^k}{l-q} = q^k \]

\[ P[\text{m} > k] = 0.02^{10} \]
Zeta is the sample space of the underlying random experiment.
S is the set of Zeta that is associated with the RV
Sx is the set of all values taken by the RV

A random variable X is a function X(s) that assigns a real number to each outcome in the sample space of an experiment.

S is the domain of X
Sx is the range of X

Flip a coin 3 times. Let X be the number of heads.
S: HHH, HHT, HTH, THH, THT, TTH, HTT, TTT
X(s): 3 2 2 2 2 1 1 0
Sx = {0, 1, 2, 3}
Exponential RV – Continuous

\[ x \in [0, \infty) \]

\[ f(x) = \lambda e^{-\lambda x}, \quad x \geq 0 \]

**Mean or Expected Value**

\[ \mathbb{E}[x] = \int_0^{\infty} x f(x) \, dx \]

The pdf specifies the probability of events like:

- "\( x \) falls in a small interval around the point \( \xi \)"

\[ P[a \leq x \leq b] = \int_a^b f(x) \, dx \]

CDF:

\[ F(x) = \begin{cases} 0 & x < 0 \\ -e^{-\lambda x} & x \geq 0 \end{cases} \]

Cumulative Distribution Function (CDF)
\[ \text{The amount of time required by a cable modem to obtain access to the upstream channel is exponentially distributed with an average of } 0.005 \text{ sec. What is the probability a customer will wait longer than 0.11 seconds?} \]

\[
E[X|x] = \frac{1}{\lambda} = 0.005 \quad \lambda = 200
\]

\[
P[X > 0.01] = \int_{0.01}^{\infty} \lambda e^{-\lambda x} \ dx
\]

\[
P[X > 0.01] = 1 - F_X(0.01)
\]

\[
= 1 - \left(1 - e^{-\lambda x}\right)
\]

\[
= e^{-\lambda x}
\]

\[
= e^{-2 \times 0.01}
\]

\[
= e^{-0.02}
\]

\[
= e^{\ln(0.1)}
\]

\[
= 0.1
\]
3 Types of RV's

- **Discrete**: A RV when cdf is a right-continuous Marsson function of $X$ with jumps at a countable set of points $x_1, x_2, x_3, \ldots$

Eq define RV $X$ to be the # of heads in 3 flips of a fair coin

$\{x\} \in \{0, 1, 2, 3\} \quad \text{range of } X$

$F_X(x) \quad \text{pdf}$

- **Continuous**: RV $X$ has a continuous cdf that can be written

$$F_X(x) = \sum_{-\infty}^{+\infty} f(T) \, \Delta T$$

- **Mixed**: RV is a RV with a CDF that has jumps on a countable set of points $x_0, x_1, x_2, \ldots$ but that also increases continuously over at least one interval of values of $X$. The cdf has the form

$$F_x(x) = p F_1(x) + (1 - p) F_2(x)$$
The pdf of a uniform RV

\[ f_x(x) = \frac{1}{b-a}, \quad a \leq x \leq b \]

\[ F_x(x) = \begin{cases} 0 & \text{if} \ x < a \\ \frac{x-a}{b-a} & \text{if} \ a \leq x < b \\ 1 & \text{if} \ x \geq b \end{cases} \]

Consider 3 regions:

1) \( x \leq a \)

2) \( a \leq x < b \) \( \Rightarrow \int_a^b \frac{1}{b-a} \, dx + \int_x^b \]

3) \( x \geq b \) \( \Rightarrow \int_a^x \frac{x-a}{b-a} \, dx \)
\[
F_x = \begin{cases} 
0 & \text{if } x < a \\
\frac{x-a}{b-a} & \text{if } a \leq x < b \\
1 & \text{if } x \geq b
\end{cases}
\]
\[ E[X] = \int_{-\infty}^{\infty} x f(x) \, dx \]

\[ = \frac{1}{b-a} \left[ \frac{1}{2} x^2 \right]_a^b - \frac{1}{b-a} \frac{1}{2} \int_a^b x^2 \, dx \]

\[ = \frac{1}{2} \frac{b^2 - a^2}{b-a} - \frac{1}{6} \frac{b^3 - a^3}{b-a} \]

\[ = \frac{1}{2} \frac{(b+a)(b-a)}{b-a} = \frac{b+a}{2} \]

\[ \frac{b-a}{b-a} = \frac{1}{2} \]