Poisson Discrete Random Variable: $X$.

- The "counting" RV $\sum_{i=1}^{n}$ models phone call demand per minute.
- $X \in \{0, 1, 2, \ldots\}$ for radioactive substances.

In general, it can model count totals (random things).

Let $X$ be the events in a Time Interval $T$ when the interval between events is exponentially distributed with mean $\lambda$.

- $X$ might be given as Rate of arrivals ($\lambda$) and a Time Interval $T$.
- The expected value of $X = \lambda$ which is $\lambda T$.

Example: Calls arrive at a telephone switch at random times with an average rate of 0.25 calls/sec. Define $X$ to represent the number of calls that arrive in a 2-second interval.

The probability function (PMF) of $X$ is:

$P_X(x) = \frac{(0.5)^x e^{-0.5}}{x!}$ for $x = 0, 1, 2, \ldots$

$P_X(x) = 0$ otherwise.

What is the probability of calls occurring in $T$?

$P_X(x=0) = \frac{(0.5)^0 e^{-0.5}}{0!} = 0.606$

$P_X(x=1) = \frac{(0.5)^1 e^{-0.5}}{1!} = 0.303$

$P_X(x=2) = \frac{(0.5)^2 e^{-0.5}}{2!} = 0.075$
\[ P_X(x) = \frac{\alpha^x e^{-\alpha}}{x!} \]

Ex. The hits at a web site in any time interval is modeled as a Poisson RV.

A site has on avg 2 hits/sec. In an interval of 0.25 sec:

What is the prob. of no hits? 

\[ \lambda = \frac{1}{4} \times 2 = 0.5 \]

Prob of no hits,

\[ P_X(0) = \frac{e^{-0.5} \cdot 0.5^0}{0!} = 0.607 \]

What is the prob. that there are no more than 2 hits in an interval of 1 sec? 

\[ P_X(\leq 2) = P_X(0) + P_X(1) + P_X(2) \]

\[ \lambda = 2 \]

\[ = e^{-2} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!} = 0.677 \]
Expect value of a discrete RV $X$

$$E[X] = \sum_{x \in S_X} xP_X(x)$$

**Example:**
- Let $X$ be a binomial RV.
- $P_X(x) = \begin{cases} 1 - p & x = 0 \\ p & x = 1 \\ 0 & \text{otherwise} \end{cases}$

$$E[X] = 0 \cdot P_X(0) + 1 \cdot P_X(1) = 0(1-p) + 1(p) = p$$

**Example:**
- Let $X$ be a Poisson RV.
- $P_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}$

**Expected value:**
$$E[X] = \sum_{x=0}^{\infty} x \cdot \frac{\lambda^x e^{-\lambda}}{x!} = \lambda$$

**Expected value using a different method:**
- Define $e^\lambda = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots$
- For $x \geq 1$, evaluate $e^\lambda$ as a limit of the sum of the terms.

**Example:**
- Let $X \sim \text{Poisson}(\lambda)$.
- $E[X] = \lambda$

**Sum of the Poisson distribution:**
- Let $X_1, X_2, \ldots$ be independent Poisson RVs with parameter $\lambda$.
- $E[X_1 + X_2 + \cdots] = \sum_{i=1}^{\infty} \lambda = \infty$
Consider a random experiment specified by the outcomes \( \xi \) (events) from \( S \), by the events defined in \( S \) (\( S_x \)), and by the probabilities of the events.

For every outcome \( \xi \in S \), assign a function of time

\[ X(t, \xi) \]

The graph of \( X(t, \xi) \) for a fixed \( \xi \) is called a realization, sample path, or path of the RP.

\( X(t, \xi_1) \)

\( X(t, \xi_2) \)
$\exists \tilde{x}(T, g)$, where $T \in T$

It is an element of an $\mathcal{F} \times I$

A RP is discrete-time if

$\{x(t, g), t \in I\}$

So a RP is an indexed family of RVs

$X(t, g)$

The realizations of the RP are sinusoids with amplitude $g$. 

$\text{RP:}$
Bivariate independent and identical distribution

\[ X \text{ is i.i.d.} \]

Binomial RV - number of successes in \( n \) Bernoulli Trials

\[
P_k = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n
\]

\[ S_X = \{ 0, 1, \ldots, n \} \]

Geometric RV

\[ S_X = \{ 0, 1, 2, \ldots \} \]

\[ P_k = p (1-p)^k \]

or can write as \( P[X = k] = p (1-p)^k \)

The probability \( P[X = k] \) decays geometrically with \( k \)

Exponential RV - time between occurrences of events

\[ X \text{ might be time in between occurrences of events} \]

\[ f(x) = P[X = x] = \int_0^\infty e^{-\lambda x} \, dx = \frac{1}{\lambda} e^{-\lambda x}, \quad x \geq 0 \]

\[ \lambda \text{ is the rate of occurrence of events} \]

\[ F(x) = P[X \leq x] = \int_0^x e^{-\lambda x} \, dx = 1 - e^{-\lambda x}, \quad x \geq 0 \]
Review of Random Processes

A R.P. on a Stochastic Process is an indexed family of random variables (RV)

A continuous-value RP is one in which the RV $X(t)$ with $t$ fixed takes on continuous values.

A RP is continuous-time if $I$ is continuous.

A RP is discrete-time if $I$ is a countable set.

The graph of function $X(t, s)$ versus $t$ for fixed $s$ is called a realization or sample path of the RP.

For fixed $t_k$ from the index set $I$, $X(t_k, s)$ is a RV.
A M.S.P. is a mathematical model of a probabilistic experiment that evolves in time and generates a sequence of numerical values.

Each numeric value in the sequence is modeled by a RV.

- a S.P. is a sequence of RVs

We might be interested in any of the following:

- Dependencies in the sequence of values generated by the process

- Long-term averaged involving the sequence of values

- The frequency of particular boundary events e.g., the probability a queue overflows
(P.P.)
The Poisson R.P. is the continuous time equivalent to the Bernoulli process.
It applies to situations where there is no natural way of dividing
Time into discrete periods.

Definition of P.P.

a) Time-homogeneity or Stationary Increments
   The prop. \(P(k, t)\) of \(k\) arrivals
   is the same for all intervals
   of the same length \(t\).

b) Independence or Independent Increments
   The \# of arrivals during a
   particular interval is independent
   of the history of arrivals
   outside this interval.

c) Small Interval Probabilities:
\[
\begin{align*}
    P(0, t) &= 1 - \lambda t + O(t) \\
    P(1, t) &= \lambda t + O_1(t) \\
    P(k, t) &= O_k(t) \text{ for } k \geq 3
\end{align*}
\]

So, the prop. of a single arrival is \(\lambda t\).
The prop. of 0 arrivals is \(1 - \lambda t\).
The prop. of >1 arrivals is \(0\).
Which allows the Poisson-Bernoulli process
to approximate the P.R.P.
Let $X_1, X_2, \ldots, X_k$ be the $k$ RVs obtained by sampling the RP $X(t, \theta)$ at times $T_1, T_2, \ldots, T_k$.

$$X_i = X(T_i, \theta), \quad X_2 = X(T_2, \theta), \quad \ldots \quad X_k = X(T_k, \theta)$$

The joint behavior of the RP at these $k$ time instants is specified by the joint CDF of the vector RV $(X_1, X_2, X_k)$.

$$F_{X_1, \ldots, X_k}(x_1, x_2, \ldots, x_k) = P[X_1 \leq x_1, X_2 \leq x_2, \ldots, X_k \leq x_k]$$

Luckily we do not have to deal with a vast collection of joint CDFs as most RPs are described by simple models.

2 Important Classes (Types) of RPs

1) A RP is said to have independent increments if for any $k$ and any choice of sampling instants $T_1 < T_2 < \ldots < T_k$, the RVs

$$X(T_2) - X(T_1), X(T_3) - X(T_2), \ldots$$

are independent RVs.

2) A RP is Markov if the future of the process is independent of the past and depends only on the present.
Examples of Discrete-Time RP.

IID RP - $X_n$ is a discrete-time RP consisting of a sequence of independent, identically distributed (i.i.d.) RVs with common CDF $F_X(x)$, mean $m$ and variance $\sigma^2$.

Mean of an IID RP

$m_X(n) = E[X_X] = m$ for all $n$ (BP)

A Bernoulli process is a sequence of $X_1, X_2, \ldots$ of independent Bernoulli RVs

$P[X_1 = 1] = P[\text{success at } 1^{st} \text{ trial}] = p$

$P[X_1 = 0] = P[\text{failure at the } 1^{st} \text{ trial}] = 1 - p$

The process exhibits independence and memorylessness.

These properties are related.

For any given time $n$, the sequence of RVs $X_{n+1}, X_{n+2}, \ldots$ (i.e. the future) is also a BP and is independent from $X_1, X_n$ (in the past).
Continuous-Time RPs

Poisson Process — Useful for modeling situations in which events occur at random instants in time at an average rate of \( \lambda \) events per second.

The \( \# \) of events \( N(\tau) \) in the interval \( [0, \tau] \) had a Poisson distribution with mean \( \lambda \tau \):

\[
P[N(\tau) = k] = \frac{e^{-\lambda \tau} \lambda^k}{k!}
\]

for \( k = 0, 1, \ldots \)

The Poisson RP had 2 important properties:

1) Independent increments —
2) Stationary increments — increments in intervals of the same length have the same distribution regardless of when the interval started.

Ex. Packets arrive at a router at a rate of 15 per minute. Find the probability that in a 1-minute period, 3 packets arrive during the first 15 seconds and 2 packets arrive during the last 15 seconds

Arrival Rate \( \lambda = 15/60 = \frac{1}{4} \) packets/second

\( P[N(0) = 3] \) and \( P[N(0) = 2] \)
The probability of interest is \[ P[N(10) = 3 \text{ and } N(60) - N(45) = 2] \]

Apply independent increments: \[ P[N(10) = 3] P[N(60) - N(45) = 2] \]

Apply Markovian increments: \[ P[N(10) = 3] P[N(60 - 45) = 2] \]

\[
= \frac{(10/4)^3 e^{-10/4}}{3!} \cdot \frac{(15/4)^2 e^{-15/4}}{2!}
\]
For a Poisson RP

Consider time $T$ between event occurrences

Suppose that the time interval $[0, T]$ is divided into $n$ subintervals of length $\delta = \frac{T}{n}$. The probability that the inter-event time $T$ exceeds $\delta$ seconds is equivalent to no event occurring in $T$ seconds (or in $n$ Bernoulli trials)

$\to 0,0

\to \frac{T}{n}$

\[ \text{If } n \text{ is large the prob. of } \geq 1 \text{ event occurrence in a subinterval is negligible compared to the prob. of observing zero or one events.} \]

A Poisson RP assumes whether or not an event occurs in a subinterval is independent of the outcomes in other subintervals

#1 allows us to assume the outcome of each subinterval can be viewed as a Bernoulli Trial.

#2 allows us to assume each Bernoulli Trial is independent.

Therefore, $p$ is an exponential with parameter $\lambda$.

Inter-event times for a $p$-form are $= \frac{\lambda}{n} \to \exp$ as $n \to \infty$, so $2 \lambda \cdot \exp = \frac{\lambda}{n}$

$p = \frac{\lambda}{n}$
- The sequence of inter-event times in a Poisson process is composed of independent random variables (RVs).

The inter-event times form a sequence of independent, identically distributed exponential RVs with mean \( \frac{1}{\lambda} \).

When applied to arrivals, we say the events occur at random. Support we are given that an arrival occurred in \([0, t]\).

Let \( X \) be the arrival time of the customer.

\[
P[X < x] = \frac{\lambda e^{-\lambda x}}{\lambda} \quad \text{for} \quad \lambda > 0, x > 0
\]

Let \( N(x) \) be the number of events up to time \( x \).

Remind definition of conditional probability:

\[
P[A \mid B] = \frac{P[A \cap B]}{P[B]}
\]

Let \( N(x) - N(x^0) \) be the increment in the interval \([x^0, x]\).

\[
P[N(x) \text{ and } N(x) - N(x^0) = 0] = P[N(x) = 1] = \frac{1}{\lambda} e^{-\lambda x}
\]

If an arrival occurs in \([0, t]\), the arrival time is uniformly distributed in the interval \([0, t]\).
A discrete-time or continuous-time RP $X(t)$ is stationary if the joint distribution of any set of samples does not depend on the placement of the time origin.

A process is wide sense stationary if its expected value is constant and its auto-correlation function depends only on the time difference.
How can we generate (simulate) a Poisson Arrival Process?

A stationary process with rate $\lambda > 0$ has the property that arrival times $A_n = t_n - t_{n-1}$ (where $n=1, 2, ...$) are i.i.d. exponential random variables with common mean $\frac{1}{\lambda}$.

We want to generate $t_1$'s.

1) generate $U \sim U(0, 1)$

2) return $t_n = -\frac{1}{\lambda} \ln(U)

for a given run. This produces a sample path $\{t_1, t_2, ..., t_n\}$.

Switch to a new stream and it will produce a different, independent sample path.

Remember: Generating an exponential random variable $t_\lambda = -\frac{1}{\lambda} \ln(U)$