Some ideas behind the feedback centralities
Counting All Paths

Simple, directed $G = (V, E)$, no loops

$$\forall i \in V \ c_K(i) = \sum_{k=1}^{\infty} \sum_{j=1}^{n} \alpha_k (A^k)_{ji}$$

The sum converges with restricted $\alpha_k$.

**Theorem.** If $A$ is the adjacency matrix of $G$, $\alpha > 0$, and $\lambda_1$ the largest eigenvalue of $A$, then

$$\lambda_1 < 1/\alpha \iff \sum_{k=1}^{\infty} \alpha^k A^k \text{ converges}$$

and $c_K = (I - \alpha A)^{-1} \cdot 1_n$.

Leo Katz “A new status index derived from sociometric analysis”
Network Flow

A flow network is given by a directed graph \( G = (V, E) \), edge capacity function \( u : E \rightarrow \mathbb{R}_{\geq 0} \), and two distinct nodes \( s, t \in V \). A flow from \( s \) to \( t \) is a function \( f : E \rightarrow \mathbb{R} \) satifying the following constraints

- **Capacity:** \( \forall e \in E : 0 \leq f(e) \leq u(e) \)

- **Balance:** \( \forall v \in V \setminus \{s, t\} : \)

\[
\sum_{e \in \Gamma^{-}(v)} f(e) = \sum_{e \in \Gamma^{+}(v)} f(e)
\]

The value of the flow \( f \) is defined as \( \sum_{e \in \Gamma^{+}(s)} f(e) - \sum_{e \in \Gamma^{-}(s)} f(e) \). Computing a flow of a maximum value is important problem. Goldberg and Tarjan solved it in \( O(nm \log(n^2/m)) \). Ford-Fulkerson theorem says that the value of a maximum s-t-flow = the capacity of minimum s-t-cut.
Vitality (robustness)

Let $\mathcal{G}$ be the set of all simple, undirected and unweighted graphs $G = (V, E)$ and $f : \mathcal{G} \rightarrow \mathbb{R}$ be any real-valued function on $G \in \mathcal{G}$. A vitality index $\mathcal{V}(G, x)$ is the difference of the values of $f$ on $G$ and on $G$ without element $x$, i.e.,

$$\mathcal{V}(G, x) = f(G) - f(G \setminus x).$$

**Max-flow Betweenness Vitality**

**Q:** How much flow must go over a vertex $i$ in order to obtain the maximum flow value? How does the objective function value change if we remove $i$ from the network?

$$c_{mf}(i) = \sum_{\substack{s, t \in V \setminus i \neq s, i \neq t \setminus f_{st} > 0}} \frac{f_{st}(i)}{f_{st}}, \text{ where } f_{st}(i) = f_{st} - \max \text{ s-t-flow in } G \setminus i$$

Examples of vitality: power grids, social networks with no leader, collaboration networks
Closeness Vitality

Wiener index of a network

\[ I_W(G) = \sum_{i,j \in V} \delta_{ij} \]

or in terms of closeness centrality

\[ I_W(G) = n \cdot \sum_{i \in V} \frac{1}{C_i} \]

Closeness vitality is defined on both vertices and edges

\[ c_{CV}(x) = I_W(G) - I_W(G \setminus \{x\}) \]

Computational problem with this vitality index?
Stress Centrality as a Vitality Index

\[ \sigma_{st}(i) \] is a number of \( s-t \) shortest paths containing \( i \)

\[ \sigma_{st} \] is a number of all \( s-t \) shortest-paths

Stress Centrality

\[ c_S(i) = \sum_{s \neq i} \sum_{t \neq i} \sigma_{st}(i) \] for nodes

\[ c_S(ij) = \sum_{s \in V} \sum_{t \in V} \sigma_{st}(ij) \] for edges

Can be interpreted as the number of shortest paths that are lost if the vertex or edge is removed from the graph. However, ... (what can be the problem?)

![Diagram](image)

**Fig. 3.9.** The figure shows that the removal of an edge can actually increase the number of shortest paths in a graph.

Homework (grads only; bonus for undergrads): 1) check at home; 2) when will the removal of an edge lead to an increase in the edge number? 3) any solution to this problem? (submit by 2/3/2014)
Current Flow

- Electrical network is an undirected, simple, connected graph $G = (V, E)$
- Conductance function $c : E \rightarrow \mathbb{R}$
- Supply function $b : V \rightarrow \mathbb{R}$ (external electrical current enters and leaves network)
- Positive $b = \text{entering current}$
- Negative $b = \text{leaving current}$
  \[ \sum_{i \in V} b(i) = 0 \]
- Direction of the current: each edge $ij$ in $E$ is oriented arbitrarily

  Function $x : \overline{E} \rightarrow \mathbb{R}$ is called current if

  \[ \sum_{ij \in \overline{E}} x_{ij} - \sum_{ji \in \overline{E}} x_{ji} = b(i) \text{ and } \sum_{ij \in C} x_{ij} = 0 \]

  for every cycle $C \subset E$. \hspace{1cm} \text{undirected}

A function $p : V \rightarrow \mathbb{R}$ is a potential if $p(i) - p(j) = x_{ij}/c_{ij}$ for all $ij \in \overline{E}$. As an electrical network $N = (G, c)$ has a unique current $x$ for any supply $b$, it also has a potential $p$ that is unique up to an additive factor.
Given edge weights $c(i)$, we define electrical network Laplacian $L$.

We can find $p$ and $b$ by solving $Lp=b$.

**Current-Flow Betweenness Centrality**

Unit $s-t$-supply $b_{st}$ is a supply of one unit that enters the network at $s$ and leaves at $t$, that is, $b_{st}(s) = 1$, $b_{st}(t) = -1$, and $b_{st}(i) = 0$ for all $i \in V \setminus \{s, t\}$.

Throughput of $i \in V$ with respect to a unit $s-t$-supply $b_{st}$ is defined as

$$
\tau_{st}(i) = \frac{1}{2} \left( -|b_{st}(i)| + \sum_{ij \ni i} |x(-ij)| \right)
$$

$$
c_{CB}(i) = \frac{1}{(n-1)(n-2)} \sum_{s,t \in V} \tau_{st}(i)
$$

How to compare different centrality concepts?

Normalization in one network

$p$-norm of the centrality vector for concept $X$

$$
||c_X||_p = \begin{cases} 
\left( \sum_{i=1}^{n} |c_{X_i}|^p \right)^{1/p} & 1 \leq p \leq \infty \\
\max_i \{|c_{X_i}|\} & p = \infty
\end{cases} \quad \Rightarrow \quad \frac{c_X}{||c_X||_p} \quad \Rightarrow \quad c_{X_i} \leq 1
$$

separation of positive and negative values of $c_X$

$$
c'_X = \begin{cases} 
c_{X_i}/\left( \sum_{j:c_{X_j}>0} |c_{X_j}|^p \right)^{1/p} & c_{X_i} > 0 \\
0 & c_{X_i} = 0 \\
c_{X_i}/\left( \sum_{j:c_{X_j}<0} |c_{X_j}|^p \right)^{1/p} & c_{X_i} < 0
\end{cases}
$$

Exercise (do not submit): Is $c'_X$ a norm? Prove or disprove.

Freeman “Centrality in social networks: Conceptual clarification”
Normalization for different networks

Point-centrality

\[ c''_{xi} = c_{xi} / \left( \max_{G \in \mathcal{G}_n} \max_{i \in V(G)} c_{xi} \right) \]

set of all graphs with \( n \) vertices

Examples

- Degree centrality = normalization by factor \((n-1)\)

- Shortest paths betweenness centrality \( c_B(i) = \sum_{s \neq i} \sum_{t \neq i} \sigma_{st}(i) / \sigma_{st} \)

  What is the upper bound (or normalization factor)?

  **Star graph, \( c_B(i) = (n-1)(n-2)/2 \)**

- Closeness centrality

  \( \forall i \in V \quad C_i = 1/l_i \), where \( l_i = \frac{1}{n} \sum_{j \in V} \delta_{ij} \), \( \delta_{ij} = \) length of \( i - j \) shortest path

  What is the upper bound (or normalization factor)? It is \( 1/(n-1) \)