Exponential Random Graphs Model

Instead of analyzing one network with fixed parameters, it is useful to consider ensembles of networks that are similar to the original.

Let us fix average values of some network properties (such as clustering and modularity). Possible property of an ensemble: values closer to the averages have higher probability. Define

\[
\sum_{G \in \mathcal{G}} \Pr(G) = 1
\]

graphs with \( n \) nodes

For network measure \( x_i, 1 \leq i \leq M (\ll 2^{n(n-1)/2}) \)

\[
\langle x_i \rangle = \sum_{G \in \mathcal{G}} \Pr(G) x_i(G)
\]

i.e., if \( \Pr(G) \) are variables then such systems do not describle the system completely.

How to choose \( \Pr(G) \)?
Best choice of probability distribution given a small number of constraints maximizes Gibbs entropy

\[ S = - \sum_{G \in \mathcal{G}} \text{Pr}(G) \ln \text{Pr}(G) \]

Maximization of entropy with Lagrange multipliers

\[
\max \quad - \sum_{G \in \mathcal{G}} \text{Pr}(G) \ln \text{Pr}(G) - \alpha (1 - \sum_{G \in \mathcal{G}} \text{Pr}(G)) - \sum_{i} \beta_{i} (\langle x_{i} \rangle - \sum_{G \in \mathcal{G}} \text{Pr}(G) x_{i}(G))
\]

Differentiate wrt \( P(G) \) of a particular \( G \)

\[- \ln \text{Pr}(G) - 1 + \alpha + \sum_{i} \beta_{i} x_{i}(G) = 0\]

or

\[ \text{Pr}(G) = \exp(\alpha - 1 + \sum_{i} \beta_{i} x_{i}(G)) \Rightarrow \text{Pr}(G) = \frac{e^{H(G)}}{Z}, \]

where \( Z = e^{1-\alpha} \) and \( H(G) = \sum_{i} \beta_{i} x_{i}(G) \) is the graph Hamiltonian.
$Z$ is solved by normalization

$$\sum_{G \in \mathcal{G}} \Pr(G) = \frac{1}{Z} \sum_{G \in \mathcal{G}} e^{H(G)} = 1$$

$\beta_i$ are solved by substituting $\Pr(G) = \frac{e^{H(G)}}{Z}$ into $\sum_{G \in \mathcal{G}} \Pr(G) x_i(G) = \langle x_i \rangle$

In general $\beta_i$ can play a role of importance coefficients.

**Practice**

If we have $\Pr(G)$ over graphs let us estimate useful quantities. For property $y$

$$\langle y \rangle = \sum_{G \in \mathcal{G}} \Pr(G) y(G) = \frac{1}{Z} \sum_{G \in \mathcal{G}} e^{H(G)} y(G)$$

Example: Fix the expected number of edges only. Then $H = \beta m$ and individual graphs appear with prob

$$\Pr(G) = \frac{e^{\beta m}}{Z}, \text{ where } Z = \sum_G e^{\beta m} \Rightarrow \text{ higher } \beta \text{ correspond to denser networks}$$
R-Mat Generator
by Chakrabarti, Zhang, Faloutsos

Initially

Choose quadrant b

Choose quadrant c

and so on

Final cell chosen, “drop” an edge here.

taken from presentation by C. Faloutsos at SIAM DM04

a = 0.4  b = 0.15

b = 0.15  c = 0.3

d = 0.15  d = 0.3
R-Mat Generator
by Chakrabarti, Zhang, Faloutsos

Communities within communities

Communities

Cross-community links

RedHat
Mandrake

Linux guys
Windows guys

taken from presentation by C. Faloutsos at SIAM DM04
Kronecker Graphs

**Definition 1 (Kronecker product of matrices)** Given two matrices $A = [a_{i,j}]$ and $B$ of sizes $n \times m$ and $n' \times m'$ respectively, the Kronecker product matrix $C$ of dimensions $(n \cdot n') \times (m \cdot m')$ is given by

$$C = A \otimes B = \begin{pmatrix}
a_{1,1}B & a_{1,2}B & \ldots & a_{1,m}B \\
a_{2,1}B & a_{2,2}B & \ldots & a_{2,m}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{n,1}B & a_{n,2}B & \ldots & a_{n,m}B
\end{pmatrix}$$

We then define the Kronecker product of two graphs simply as the Kronecker product of their corresponding adjacency matrices.

**Definition 2 (Kronecker product of graphs (Weichsel, 1962))** If $G$ and $H$ are graphs with adjacency matrices $A(G)$ and $A(H)$ respectively, then the Kronecker product $G \otimes H$ is defined as the graph with adjacency matrix $A(G) \otimes A(H)$.

from paper [Kronecker Graphs: An approach to modeling networks](https://doi.org/10.1109/TPAMI.2010.184) by J. Leskovec, D. Chakrabarti, J. Kleinberg, C. Faloutsos, Z. Ghahramani
Example of Kronecker multiplication: Top: a “3-chain” initiator graph and its Kronecker product with itself. Each of the $X_i$ nodes gets expanded into 3 nodes, which are then linked using Observation 1. Bottom row: the corresponding adjacency matrices. See figure 2 for adjacency matrices of $K_3$ and $K_4$. 

from paper *Kronecker Graphs: An approach to modeling networks* by J. Leskovec, D. Chakrabarti, J. Kleinberg, C. Faloutsos, Z. Ghahramani
Figure 2: Adjacency matrices of $K_3$ and $K_4$, the $3^{rd}$ and $4^{th}$ Kronecker power of $K_1$ matrix as defined in Figure 1. Dots represent non-zero matrix entries, and white space represents zeros. Notice the recursive self-similar structure of the adjacency matrix.

Figure 3: Two examples of Kronecker initiators on 4 nodes and the self-similar adjacency matrices they produce.

from paper [Kronecker Graphs: An approach to modeling networks](https://doi.org/10.1145/1273496.1273577) by J. Leskovec, D. Chakrabarti, J. Kleinberg, C. Faloutsos, Z. Ghahramani
Percolation and Network Resilience

Percolation is a process of removing some fraction of network’s nodes with adjacent edges. (more precisely site/link/cluster percolation)

- models real-life phenomena such as router failure, immunization of people, and disasters
- the process is parameterized by occupation probability $\phi$
- **Percolation transition**: when $\phi$ is large there is a giant component but as $\phi$ is decreased then gc breaks into many small components or clusters (similar to phase transition in Poisson random graphs with gc→sc)
Percolation and Configuration Model

Consider a configuration model with
- degree distribution $p_k$
- occupation probability $\phi$

consider node $i$ which

- can belong to gc, i.e., connected to it through some $j \in N(i)$
- is not in gc (i.e., not connected to it via any of $N(i)$) - define the avg probability of it $u^k$, where $k = \text{deg}(i)$ and $u$ is the same prob for one particular neighbor

- avg probability of not being in gc

$$\sum_k p_k u^k = g_0(u), \text{ where } g_0(z) = \sum_k p_k z^k \text{ or } \Pr[i \in \text{gc}] = 1 - g_0(u)$$

- total fraction of nodes in gc when percolation is running

$$S = \phi(1 - g_0(u))$$