Appendices to “Implicit Crowds: Optimization Integrator for Robust Crowd Simulation”

A TIME TO COLLISION

Let two circular agents, i and j, have current positions $\mathbf{x}_i$ and $\mathbf{x}_j$, current velocities $\mathbf{v}_i$ and $\mathbf{v}_j$, and radii r_i and r_j respectively. We predict collisions by assuming a linear extrapolation of the agents' positions based on their current velocities, and determining if there is a time $t \geq 0$ at which the corresponding discs of the two agents centered at $\mathbf{x}_i + \mathbf{v}_i t$ and $\mathbf{x}_j + \mathbf{v}_j t$ intersect.

Formally, the time to collision, τ, between the two agents is defined as the earliest time t at which their extrapolated discs collide,

$$\tau = \tau(\mathbf{x}_{ij}, \mathbf{v}_{ij}, r) = \min\{t \geq 0 \mid \|\mathbf{x}_{ij} + \mathbf{v}_{ij} t\| \leq r\}. \tag{26}$$

Here τ is a function of the relative displacement, $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$, between the two agents, their relative velocity, $\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$, and their combined radius $r = r_i + r_j$.

Given equation (26), τ can be computed explicitly as the smallest positive root of $\|\mathbf{x}_{ij} + \mathbf{v}_{ij} t\|^2 = r^2$, which is a quadratic equation in t. By expanding and rearranging, we obtain

$$at^2 + 2bt + c = 0 \tag{27}$$

where

$$a = \|\mathbf{v}_{ij}\|^2, \tag{28}$$

$$b = \mathbf{x}_{ij} \cdot \mathbf{v}_{ij}, \tag{29}$$

$$c = \|\mathbf{x}_{ij}\|^2 - r^2. \tag{30}$$

Then,

$$\tau = \frac{-b - \sqrt{b^2 - ac}}{a} \tag{31}$$

In case there is no solution or the solution is not positive, we take $\tau = \infty$, which leads to a zero interaction potential.

B CONTINUOUS TIME TO COLLISION

The relative velocity, $\mathbf{v}_{ij}$, between two given agents can be decomposed into two orthogonal components, v_p and v_t, where v_p is the component of $\mathbf{v}_{ij}$ projected along the unit direction -$\hat{\mathbf{x}}_{ij}$, and v_t is its tangential counterpart:

$$v_p = -\mathbf{v}_{ij} \cdot \hat{\mathbf{x}}_{ij} \tag{32}$$

$$v_t = -\mathbf{v}_{ij} \cdot \hat{\mathbf{x}}_{ij}^{\perp}. \tag{33}$$

Then, the coefficients a and b of the quadratic equation (27) can be written as

$$a = \|\mathbf{v}_{ij}\| = v_p^2 + v_t^2, \tag{34}$$

$$b = \mathbf{x}_{ij} \cdot \mathbf{v}_{ij} = -v_p \|\mathbf{x}_{ij}\|. \tag{35}$$

Assuming that $v_p \geq 0$, that is, the agents are approaching each other, the time to collision is given as in equation (31). Unfortunately, τ is not continuously defined but jumps between finite and infinite values as the value of v_t changes, resulting in an interaction potential that jumps from zero to a finite nonzero value (see Figure 2b in the main text). To address this issue, we consider the reciprocal of the time to collision, $\sigma = \tau^{-1}$, which is easier to work with. As with τ, σ is a function of $\mathbf{x}_{ij}$, $\mathbf{v}_{ij}$, and r. Our goal is to define a continuous approximation to this function, $\bar{\sigma}$.

The reciprocal time to collision satisfies the quadratic equation $a + 2b\sigma + c\sigma^2 = 0$, obtained by dividing equation (27) by t^2. In particular, it is the largest positive root,

$$\sigma = \frac{-b + \sqrt{b^2 - ac}}{c}. \tag{36}$$

By substituting equations (30), (34), and (35) to equation (36), it follows that

$$\sigma = \frac{v_p \|\mathbf{x}_{ij}\| + \sqrt{(r\, v_p)^2 - (\|\mathbf{x}_{ij}\|^2 - r^2)\, v_t^2}}{\|\mathbf{x}_{ij}\|^2 - r^2}. \tag{37}$$

If the quadratic equation has no positive solutions, there is no collision and we define $\sigma = 0$.

To smooth over the discontinuity, we focus attention on how σ varies depends on v_t, and therefore treat it as a function $\sigma(v_t)$ holding the other parameters v_p, $\mathbf{x}_{ij}$, and r constant. As can be seen from equation (37), σ assumes a maximum value when $v_t = 0$. As σ is symmetric in v_t, without loss of generality we focus on the case where $v_t \geq 0$. The largest v_t at which σ is nonzero is

$$v_t^{\max} = \frac{r\, v_p}{\sqrt{\|\mathbf{x}_{ij}\|^2 - r^2}}. \tag{38}$$

We define $\bar{\sigma}$ by first selecting a velocity $v_t^* \in [0, v_t^{\max}]$, namely

$$v_t^* = \sqrt{1 - \epsilon^2}\, v_t^{\max}. \tag{39}$$

Here, $\epsilon \in (0, 1)$ is a constant controlling the amount of smoothing, allowing us to pick a specific σ value along the curve between the minimum and maximum σ's. See Figure 3a in the main text. Then, for any $v_t \leq v_t^*$, we keep $\bar{\sigma} = \sigma$, while the smoothed $\bar{\sigma}$ for $v_t > v_t^*$ is computed using linear extrapolation from v_t^*:

$$\bar{\sigma}(v_t) = \begin{cases} \sigma(v_t) & \text{if } v_t \leq v_t^*, \\ \max\left(0, \sigma(v_t^*) + (v_t - v_t^*)\, \sigma'(v_t^*)\right) & \text{if } v_t > v_t^*. \end{cases} \tag{40}$$

The value and slope of σ at v_t^* are given by

$$\sigma(v_t^*) = \frac{(\|\mathbf{x}_{ij}\| + \epsilon\, r)\, v_p}{\|\mathbf{x}_{ij}\|^2 - r^2}, \tag{41}$$

$$\sigma'(v_t^*) = -\frac{\sqrt{1 - \epsilon^2}}{\epsilon \sqrt{\|\mathbf{x}_{ij}\|^2 - r^2}}. \tag{42}$$

Equation 40 results in a reciprocal time-to-collision function that is continuous everywhere and differentiable away from 0, as compared to the non-smooth τ formulation of equation (31).

C POWER-LAW FIT

The original formulation of the R_{TTC} potential in Karamouzas et al. [2014] defines the associated force as the gradient with respect to positions, $\mathbf{f}_{ij} = -\nabla_1 R_{\text{TTC}}(\mathbf{x}_{ij}, \mathbf{v}_{ij})$, instead of the velocity gradient we use. The effect of this difference is primarily in the exponent of the power law, as can be seen by considering the one-dimensional case:

$$\frac{\partial}{\partial x_{ij}} R_{\mathrm{TTC}}(x_{ij}, v_{ij}) = k\tau^{-(p+1)} e^{-\tau/\tau_0} \left(\frac{\tau/\tau_0 + p}{v_{ij}} \right), \tag{43}$$

$$\frac{\partial}{\partial v_{ij}} R_{\mathrm{TTC}}(x_{ij}, v_{ij}) = k\tau^{-p} e^{-\tau/\tau_0} \left(\frac{\tau/\tau_0 + p}{v_{ij}} \right) \tag{44}$$

(see Figure 13). This suggests that to obtain comparable behavior to the original model with exponent p, it may be preferable to use exponent $p + 1$ in our method. Additionally, interactions resulting from goal forces, repulsive potentials, and approximations due to smoothing will all impact the effective power law distribution observed during simulation. In practice, we find an exponent of 3.25 most closely reproduces the results reported in Karamouzas et al. [2014], though a wide range of values all result in visually plausible behavior.

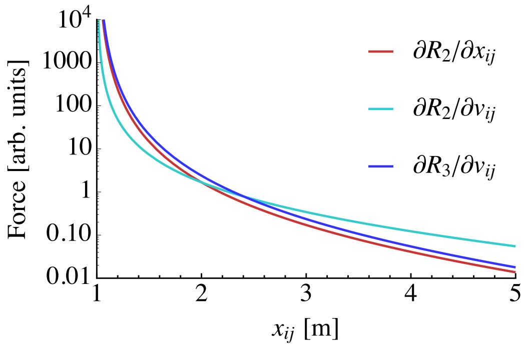

Fig. 13. Comparison of $\partial R/\partial x$ and $\partial R/\partial v$ forces in a 1D scenario with $v_{ij} = -1$ and $r = 1$, where the potential R is defined as in equation (15) and the number in the subscript denotes the exponent p of the time-to-collision power law. With velocity-based gradients, the TTC force increases more slowly near collisions (compare $\partial R_2/\partial x$ and $\partial R_2/\partial v$) unless the exponent is increased.