The Subtraction Game

Need: A supply of counters.

1. **Introduce this game.** Create a pile of 10 counters. **Players take turns removing counters.** The person who gets the last counter is the winner. Each move a player can take 1, 2 or 3 counters.

   After students play with each other a couple of times, ask whether they would prefer to be the First Player (the one who moves first) or the Second Player. If the First Player, ask what their best first move is.

   At least some students should notice that a pile with a multiple of 4 is important. For such a pile the Second Player can win by “doing the opposite” of the other player, always returning the pile to a multiple of 4.

2. **Change the game:** a player can take only 1 or 3 counters at a turn. Let them play, or maybe they might see a similar analysis immediately. Ask them which player they want to be this time.

   The answer is that with a pile of 10 coins, the Second Player still wins. Indeed, this is a trick game. Because the number of coins is even, the second player wins no matter what moves either player makes! First Player leaves an odd number, Second Player leaves an even number. And zero is even.

3. **Change the game again.** One example is that the legal moves are to remove 1, 3 or 4 counters.

   For this game it is harder to determine which player is better. But one can determine this by building a table upwards of who wins for a given pile size:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fir</td>
<td>Sec</td>
<td>Fir</td>
<td>Fir</td>
<td>Fir</td>
<td>Fir</td>
<td>Sec</td>
<td>Fir</td>
<td>Sec</td>
<td>Fir</td>
<td></td>
</tr>
</tbody>
</table>

   The First Player wins the pile of 10, by removing either 1 or 3.

4. **Take-Away Idea:** Almost any variation is interesting. For example, one can switch the game around by saying that the person who takes the last counter is the loser. Note that this usually considerably changes the analysis.