Exercises for Chapter 9

I1. Compute \(a \cdot b, a \cdot c,\) and \(b \cdot c,\) when \(a = (1, -1, 1, -1), b = (-2, -2, 7, \pi),\) and \(c = (0, 0, 1, 0).\)

I2. Consider \(v = (4, 7).\)
   (a) Give a unit vector in the same direction as \(v.\)
   (b) Give a unit vector orthogonal to \(v.\)

I3. For each of the following triples, determine \(k\) such that it is orthogonal.
   (a) \((1, 0, 0), (0, 2, 1), (0, 1, k)\)
   (b) \((1, 2, -2), (2, 3, 4), (0, 3, k)\)
   (c) \((k, k, 0, 0), (k, 0, k, 0), (0, 0, k, k)\)

I4. Find a \(4 \times 4\) matrix all of whose entries are \(\pm 1\) such that the columns are pairwise orthogonal.

I5. A matrix \(P\) is defined to be a projection matrix if \(P^2 = P\) and \(P = P^T.\)
   (a) Show that every eigenvalue of any projection matrix is 0 or 1.
   (b) Show that if \(U\) is an invertible matrix, then \(P_U = U(U^T U)^{-1} U^T\) is a projection matrix.
   (c) Show that \(\text{proj}_y(U) = P_U y.\) That is, \(P_U\) is the matrix transform that projects \(y\) onto the columns of \(U.\)

I6. (a) Show that every orthonormal matrix \(U\) has determinant \(\pm 1.\)
   (b) Give an example \(U\) that has determinant 1 and contains no zeroes.
   (c) Give an example \(U\) that has determinant \(-1\) and contains no zeroes.

I7. Prove that if the columns of the square matrix \(F\) are orthonormal, then so are the rows.
   (Hint: consider \(F^{-1}.\))

I8. Use Gram-Schmidt to produce an orthonormal basis for the space spanned by \((1, 1, 1, 0, 0),\)
   \((0, 1, 1, 0)\) and \((0, 0, 1, 1, 1).\)
Some Solutions

I1. \( \mathbf{a} \cdot \mathbf{b} = 7 - \pi; \mathbf{a} \cdot \mathbf{c} = 1; \mathbf{b} \cdot \mathbf{c} = 7 \).

I2. (a) \((4/\sqrt{65}, 7/\sqrt{65})\)  
   (a) \((7/\sqrt{65}, -4/\sqrt{65})\)

I3. (a) \( k = -2 \)  
   (b) DNE  
   (c) \( k = 0 \)

I4. For example

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}
\]

I5. (a) Assume \( P \mathbf{v} = \lambda \mathbf{v} \) where \( \mathbf{v} \neq \mathbf{0} \). Then \( P^2 \mathbf{v} = \lambda^2 \mathbf{v} \). But we are given that \( P^2 = P \). Therefore \( \lambda^2 = \lambda \). This implies that \( \lambda \) is 0 or 1.  
   (b) \( P_U^2 = (U(U^TU)^{-1}U^T)(U(U^TU)^{-1}U^T) = U(U^TU)^{-1}(U^TU)(U^TU)^{-1}U^T = P_U \).  
   Similarly, \( P_U^T = P_U \).  
   (c) ADD ME

I6. (a) Since \( U^TU = I \), we have \( \det \mathbf{U} \cdot \det \mathbf{U} = 1 \).  
   (b) \( \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \)  
   (b) \( \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \)

I7. Since columns orthonormal, we have \( F^TF = I \). This means that \( F^{-1} = F^T \). So we have \( FF^T = I \) too. That is, the rows are orthonormal.

I8. \( \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \frac{1}{\sqrt{15}} \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}, \text{ and } \frac{1}{\sqrt{40}} \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \)