Exercises for Chapter 1

A1. Solve the following systems.

\[
\begin{align*}
  s + t &= 5 \\
  x - y + z &= 8 \\
  2s + t &= 1 \\
  2x - y &= 4 \\
  3y &= -6
\end{align*}
\]

A2. For what value(s) of \( \beta \) is the system \( x + \beta y = 1 \) and \( x - y = \beta \) consistent?

A3. Consider this matrix

\[
B = \begin{bmatrix}
  0 & 1 & 0 & 0 & 0 & -3 \\
  0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 & -5 \\
  0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(a) Is matrix \( B \) in echelon form? Explain.

(b) Is matrix \( B \) in reduced row echelon form? Explain.

A4. Describe geometrically the solution to the linear equation \( x + y + z = 1 \).

A5. What’s the maximum possible number of nonzero entries in a 3 \times 3 matrix in reduced row echelon form? Justify your answer.

A6. Find the general solution of the systems given by the following augmented matrices.

(Assume variables are \( x_1, x_2, \ldots \))

\[
\begin{bmatrix}
  1 & 2 & \cdots & -3 & | -1 \\
  4 & -2 & \cdots & 3 & | 0 \\
  0 & 0 & \cdots & 0 & | 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & 0 & 0 & 2 & | -12 \\
  0 & 2 & 3 & 0 & | 3
\end{bmatrix}
\]

A7. Consider \( n \) numbers \( x_1, x_2, \ldots, x_n \) laid out on a circle and some value \( \alpha \). Consider the requirement that every number equals \( \alpha \) times the sum of its two neighbors. For example, if \( \alpha \) were zero, this would force all the numbers to be zero.

(a) Show that, no matter what \( \alpha \) is, the system has a solution.

(b) Show that if \( \alpha = \frac{1}{2} \), then the system has a nontrivial solution.

(c) Show that if \( \alpha = -\frac{1}{2} \), then there is a nontrivial solution if and only if \( n \) is even.
Some Solutions

A1. $s = -4, \ t = 9$
    $x = 1, \ y = -2, \ z = 5$

A2. We can solve for $x$ and $y$ except when $\beta = -1$.

A3. (a) Yes: we have zeroes below the first nonzero in each row
    (b) Yes: the pivots are 1 and have zeroes above them

A4. A plane.

A5. Four. In the case that there are pivots in first two columns only.

A6. $x_1 = 2/5 + x_3, \ x_2 = -7/10 + x_3$
    $x_1 = 42 - x_3, \ x_5 = -27$

A7. (a) All $x_i$ zero provides a solution
    (b) All $x_i$ the same value
    (c) Alternate +1 and −1. (But why impossible if $n$ odd?)