6 Vector Spaces and Subspaces

6.1 Sets Closed Under Operations

A set $S$ is **closed** under some operation if applying that operation to elements of $S$ always produces an element of $S$.

**Example.** The integers are closed under addition: adding two integers always produces an integer. The integers are also closed under subtraction and multiplication, but not division.

**Example.** The set of positive real numbers is not closed under subtraction: for example, $2 - \pi$ is not positive. The set is closed under addition, multiplication, division, and exponentiation.

**Fact 6.1** The set of solutions to the homogeneous equation $Ax = 0$ is closed under both addition and scalar multiplication.

**Proof.** Assume $x$ and $x'$ are solutions. Then $A(x + x') = Ax + Ax' = 0 + 0 = 0$, so that $x + x'$ is a solution. Similarly, $A(cx) = c(Ax) = c0 = 0$, so that $cx$ is a solution. 

6.2 Vector Spaces

A **vector space** is a collection of objects (which we call vectors) with operations addition and scalar multiplication defined that obey the usual vector laws.

The full list of laws, also known as the **axioms** of a vector space, vector space $V$ is given on the next page. In short, they say that addition and scalar multiplication behave like they do for ordinary vectors.

More specifically, the axioms start with the requirement that the space is closed under the two operations: that is, if you take two objects in the space and add them, the sum is in the space; if you take an object in the space and scale it by some real number, the scaled object is in the space. Other axioms include that:
addition is commutative, associative, and has negation;  
> the 0 vector and 1 scalar behave as identities; and  
> addition and scalar multiplication distribute.

**Example.** $\mathbb{R}^n$, with addition and scalar multiplication as we’ve been doing, is a vector space.

**Example.** $\mathbb{P}_n$, the set of all polynomials of degree at most $n$, is a vector space. This follows since, for example, if one adds two polynomials the result is a polynomial, and its degree cannot be larger than both summands, and thus $\mathbb{P}_n$ is closed under addition. The zero of the space is the 0 polynomial. More generally, $\mathbb{P}$, the set of all polynomials, is a vector space.

**Example.** $C[t]$, the set of all continuous functions in variable $t$ with domain $\mathbb{R}$, is a vector space. One adds and scales functions just as in calculus.

**Example.** $M_n$, the set of all $n \times n$ matrices, is a vector space.

Here are the promised axioms of a vector space $V$:

**AXIOMS** For all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $V$ and all (real) scalars $c$ and $d$:

1) The sum $\mathbf{u} + \mathbf{v}$ is in $V$
2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4) There is a vector 0 such that $\mathbf{u} + 0 = \mathbf{u}$
5) There is a vector $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = 0$
6) The scalar multiple $c\mathbf{u}$ is in $V$
7) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9) $c(d\mathbf{u}) = (cd)\mathbf{u}$
10) $1\mathbf{u} = \mathbf{u}$

It should be noted that what we have defined as a vector space is sometimes called a **real vector space** because the scalars are restricted to being real numbers. A **complex vector space** would be one where the scalars are allowed to be complex numbers.
6.3 Subspaces

A subspace of a vector space $V$ is defined to be a subset $S$ of $V$ that is a vector space in its own right, using the same operations.

Since $S$ uses the same operations, most of the axioms are immediately satisfied. There are three remaining requirements that need to be checked to establish that $S$ is a subspace:

**ALGOR** Conditions for $S$ to be Subspace.

1. $S$ contains the zero vector;
2. $S$ is closed under addition;
3. $S$ is closed under scalar multiplication.

**Example.** The degree-bounded polynomial space $\mathbb{P}_n$ is a subspace of the space $\mathbb{P}$ of all polynomials. And $\mathbb{P}$ is a subspace of the space $C[t]$ of continuous functions. The set of continuous functions such that $\int_{-\infty}^{\infty} f(t) \, dt = 0$ is a subspace of $C[t]$.

**Example.** The set containing just the zero-vector is always a subspace.

The whole space is always a subspace of itself.

**Fact 6.2** Every subspace of $\mathbb{R}^3$ is either $\{0\}$, a line through the origin, a plane through the origin, or the space itself.

**Example.** Let $T$ be the set of points $(x,y)$ in $\mathbb{R}^2$ such that $|x| = |y|$. This is not a subspace. While $T$ satisfies Conditions (0) and (2) (check!), it is not closed under addition: for example $(2,-2)$ and $(3,3)$ are in $T$ but their sum $(5,1)$ is not.

**Example.** Let $U$ be the set of points $(x,y)$ in $\mathbb{R}^2$ such that $x,y \geq 0$. This is not a subspace. While $U$ satisfies Conditions (0) and (1) (check!), it is not closed under scalar multiplication: for example $(2,3)$ is in $U$, but scaling by $-1$ produces $(-2,-3)$, which is not in $U$.

One useful fact is the following:

**Fact 6.3** If $S$ is a set of vectors, then Span $S$ is a subspace.

**Proof.** Assume that $S$ is finite. (The same proof works for infinite $S$; one just has to use different notation.) Say $S = \{v_1, \ldots, v_k\}$. Then $0$ is the linear combination $0v_1 + \ldots + 0v_k$. If $x = a_1v_1 + \ldots + a_kv_k$ and $y = b_1v_1 + \ldots + b_kv_k$, then $x + y =$
Also, \( c \mathbf{v} = (c a_1) \mathbf{v}_1 + \ldots + (c a_k) \mathbf{v}_k \). So \( \text{Span}(S) \) contains \( \mathbf{0} \), and is closed under addition and scalar multiplication. That is, \( S \) is a subspace.

6.4 The Three Matrix Spaces

For a matrix \( A \), we define three fundamental sets as follows.

- **The null space** of matrix \( A \), denoted \( \text{Nul} A \), is the set of all solutions to the homogeneous system \( A \mathbf{x} = \mathbf{0} \). That is, all vectors mapped to \( \mathbf{0} \) by the matrix transform \( \mathbf{x} \mapsto A\mathbf{x} \).

- **The column space** of matrix \( A \), denoted \( \text{Col} A \), is the set of all linear combinations of columns of \( A \).

- **The row space** of matrix \( A \), denoted \( \text{Row} A \), is the set of linear combinations of rows of \( A \).

All three of these are vector spaces:

**Fact 6.4** If \( A \) is an \( m \times n \) matrix, then
(a) \( \text{Nul} A \) is a vector space, and is a subspace of \( \mathbb{R}^n \).
(b) \( \text{Col} A \) is a vector space, and is a subspace of \( \mathbb{R}^m \).
(c) \( \text{Row} A \) is a vector space, and is a subspace of \( \mathbb{R}^n \).

**Proof.** Parts (b) and (c) follow from the earlier observation that any span is a subspace (Fact 6.3). To prove (a), we need to check the three conditions. The two closure conditions were noted in Fact 6.1. Further, the zero vector \( \mathbf{0} \) is in the null space, since \( A\mathbf{0} = \mathbf{0} \). ◊

Our earlier discussion about row operations showed that:

**Fact 6.5** If two matrices are row equivalent, then they have the same row space.

6.5 Linear Transforms

A **linear transform** \( T \) is a function from one vector space to another vector space. It is required to obey the two rules:
For all vectors $\mathbf{u}$ and $\mathbf{v}$ in the domain and all reals $c$, it holds that

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$, and
2. $T(c\mathbf{u}) = cT(\mathbf{u})$.

That is, a linear transform has the property: if you add first and then transform, you get the same result as if you transform first and then add. A similar statement can be made about scaling. One can easily check that matrix multiplication obeys these two rules; that is:

**Fact 6.6** Every matrix transform is a linear transform.

The null space of a linear transform is the set of all vectors that are mapped to 0; it is often called the kernel.

**Example.** For example, differentiation is a linear transform from the polynomial space $\mathbb{P}_n$ to the polynomial space $\mathbb{P}_{n-1}$. Its kernel is the set of all constants.

It can be shown that:

**Fact 6.7** For any linear transform:

1. The kernel is a subspace of the domain.
2. The range is a subspace of the codomain.

**Practice**

6.1. Consider the following subsets of $\mathbb{R}^3$. Explain why each is is not a subspace.

(a) The points in the $xy$-plane in the first quadrant.
(b) All integer solutions to the equation $x^2 + y^2 = z^2$.
(c) All points on the line $x + z = 5$.
(d) All vectors where the three coordinates are the same in absolute value.

6.2. In each of the following, state whether it is a vector space. Justify your answer.

(a) the set of all polynomials with degree exactly 1
(b) the set of all $2 \times 2$ matrices with determinant 2
(c) the set of all diagonal $3 \times 3$ matrices
(d) the set of all vectors in $\mathbb{R}^4$ whose entries sum to 0.
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(e) the set of all antiderivatives of $f(x) = x^5$

6.3. Show that for any linear transform $T$ it holds that $T\mathbf{0} = \mathbf{0}$.

Solutions to Practice Exercises

6.1. (a) Not closed under scalar multiplication: $(1, 1, 0)$ in but not $-1(1, 1, 0)$
     (b) Not closed under scalar multiplication: $(3, 4, 5)$ in but not $\frac{1}{2}(3, 4, 5)$
     (c) Does not contain zero.
     (d) Not closed under addition: $(1, 1, 1)$ and $(1, 1, -1)$ in but not their sum $(2, 2, 0)$.

6.2. (a) No; does not include zero
     (b) No; does not include zero
     (c) Vector space
     (d) Vector space
     (e) No; does not include zero

6.3. One way to see this is that $T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0})$ by simplification, but $T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$ by the rules of a linear transform. So $T(\mathbf{0}) + T(\mathbf{0}) = T(\mathbf{0})$ which means that $T(\mathbf{0}) = \mathbf{0}$.  