5 Determinants

We give first a general definition of a determinant. For calculations, however, the formulas and results in the later sections are more useful. (And indeed, some books use those formulas as the definition.)

5.1 Introduction to Determinants

Determinants are defined for square matrices. The determinant of square matrix \( A \) is denoted \( \det A \) or indicated by the use of vertical lines replacing the square brackets of the matrix.

Before we give the gory definition, let us note that the determinant of a \( 2 \times 2 \) matrix has a famous formula:

\[
\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left| \begin{array}{cc} a & b \\ c & d \end{array} \right| = ad - bc
\]

We saw this expression earlier in the formula for the inverse of a \( 2 \times 2 \) matrix. Indeed, like in the \( 2 \times 2 \) case, we will see that the determinant being nonzero captures when a matrix is invertible.

5.2 A Definition of Determinant

The determinant of a general matrix can be defined in terms of signs and transversals.

\( \blacksquare \) A transversal of an \( n \times n \) matrix is the product of \( n \) entries, one from each row and column.

For example, in the \( 2 \times 2 \) case above, the transversals are \( ad \) and \( bc \). A transversal can be specified by listing in order the columns used: for example the \( bc \) transversal is 21. This listing is also called a permutation. For another example, in a \( 5 \times 5 \) matrix, the permutation \( \pi = 32451 \) means the product \( a_{13}a_{22}a_{34}a_{45}a_{51} \).

\( \blacksquare \) The sign of a permutation \( \pi \) is \( (-1)^k \), where \( k \) is the number of interchanges (swapping two elements) needed to change \( \pi \) to being ordered.
Example. For example, permutation $12$ has sign $+1$ (since $k = 0$); the permutation $21$ has sign $-1$ (since $k = 1$).

Some mathematics that we omit is the perhaps surprising fact that: for a specific permutation $\pi$, though there might be many $k$ that work, they are either all even or all odd, and so $(-1)^k$ is the same no matter the choice of $k$.

Finally we can give the definition of determinant:

The determinant of matrix $A$ is defined as the sum of signed transversals:

$$\det A = \sum_{\text{perm. } \pi} \text{(sign of } \pi) \times \text{(transversal for } \pi)$$

where the sum is taken over all possible permutations $\pi$.

Note that this expression agrees with the formula for the determinant of a $2 \times 2$ matrix above.

Though we will usually not evaluate determinants using this formula, it does reveal some properties. For example, if every entry in some row is zero, then every transversal is zero, and so:

**Fact 5.1** If matrix $A$ has an all-zero row or column, then $\det A = 0$.

Here is another useful property:

**Fact 5.2** The determinant of a triangular matrix is the product of the diagonal entries.

**Proof.** Every transversal except the main diagonal is guaranteed to contain a 0-entry and thus be 0. The diagonal comes from the “ordered” permutation, which has positive sign (since $k = 0$).

In particular, the determinant of the identity matrix $I$ is $\det I = 1$.

### 5.3 Recursive Formula: Cofactor Expansion

It can be shown that the definition of determinant implies the following formula.
Fact 5.3 Assume $A$ is an $n \times n$ matrix. Let $A_{ij}$ denote the matrix formed by removing row $i$ and column $j$. Then expansion across the first row of $A$ gives the formula

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{1+n} a_{1n} \det A_{1n}$$

The formula is called recursive because it gives the value in terms of the value for smaller versions of the same problem. The idea behind the proof of the formula is that if one considers each transversal of $A$ containing $a_{11}$, then it corresponds to a transversal of $A_{11}$; and thus the contribution to the overall determinant involving $a_{11}$ is given by $a_{11} \det A_{11}$.

Example. Let $B$ be the following matrix:

\[
\begin{pmatrix}
3 & 6 & 0 \\
2 & 7 & -1 \\
0 & 4 & -8
\end{pmatrix}
\]

Then $B$ has determinant

$$3 \begin{vmatrix} 7 & -1 \\ 4 & -8 \end{vmatrix} - 6 \begin{vmatrix} 2 & -1 \\ 0 & -8 \end{vmatrix} + 0 \begin{vmatrix} 2 & 7 \\ 0 & 4 \end{vmatrix} = 3 \times (-52) - 6 \times (-16) + 0 = -60.$$  

One can expand across other rows, but note that the sign is always $(-1)^{i+j}$. Each term $C_{ij} = (-1)^{i+j} A_{ij}$ in the expansion is called a cofactor.

5.4 Properties of Determinants

It turns out that:

Fact 5.4 A square matrix is invertible if and only if its determinant is not zero.

This is equivalent to saying that the determinant is zero if and only if the columns (and rows) are linearly dependent.

This result is a consequence of the more general fact that the elementary row operations do not change the determinant much. We omit the proof of the following:
Fact 5.5

Adding a multiple of a row to another row does not change $\det$. Interchanging two rows flips the sign of $\det$. Multiplying a row by a scalar does the same to $\det$.

By the result about triangular matrices earlier, the above result gives us another method to calculate the determinant:

**ALGOR** Assume $A$ is an $n \times n$ matrix. If one obtains pivots in every row/column when reducing $A$ to echelon form, without using an interchange, then the determinant of $A$ is the product of the pivots.

**Example.** Consider the matrix $B$ from earlier.

$$
\begin{bmatrix}
3 & 6 & 0 \\
2 & 7 & -1 \\
0 & 4 & -8
\end{bmatrix}
$$

This reduces to

$$
\begin{bmatrix}
3 & 6 & 0 \\
0 & 3 & -1 \\
0 & 0 & -20/3
\end{bmatrix}
$$

without interchanges. Thus the determinant of $B$ is $-60$.

Other properties include the following. We omit the proofs:

**Fact 5.6**

$\det(A^T) = \det A$

$\det(AB) = (\det A)(\det B)$

### 5.5 Applications of Determinants

Though we don’t give it or use it, a famous idea is called Cramer’s rule. This says that $Ax = b$ has solution by ratio of determinants if $A$ is invertible. There is similarly a formula for $A^{-1}$.

There is also a geometric interpretation of the determinant. The volume of a box whose sides are vectors is given by the absolute value of the associated determinant. For example, the area of the parallelogram determined by vectors $(x_1, y_1)$ and $(x_2, y_2)$ is $|x_1y_2 - x_2y_1|$.
**Fact 5.7** In $\mathbb{R}^2$, if one applies a matrix transform $M$ to some shape, then the area of the shape changes by a factor of $\det M$.

A similar result holds for the change in volumes under matrix transforms in $\mathbb{R}^3$.

**Practice**

5.1. Calculate the determinants of the following matrices using cofactors.

\[
C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 5 & 4 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 4 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad L = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 2 & 0 \\ 4 & 1 & 5 \end{bmatrix}
\]

5.2. Calculate the determinants of the matrices in the previous question using row reduction.

5.3. Suppose $A$ is a $3 \times 3$ matrix such that $\det A = 3$. Give the determinant of:

(a) $A^T$

(b) $(A^2)^{-1}$

(c) The matrix that results if one takes $A$ and replaces the 2nd row by the sum of the 1st and 3rd rows.

(d) The matrix that results if one takes $A$ and increases the 2nd row by the sum of the 1st and 3rd rows.

(e) $A + A$.

(f) $A + A^T$.

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**Solutions to Practice Exercises**

5.1. $\det C = 30$

$\det D = 9$

$\det L = -10$

5.2. $\det C = 30$

$\det D = 9$

$\det L = -10$
5.3. (a) 3
   (b) \( \frac{1}{5} \)
   (c) 0
   (d) 3
   (e) 24
   (f) Could be anything.