Matrix Equations
Matrix Equations

**Fact.** The matrix equation $Ax = b$ has a solution if and only if $b$ is a linear combination of the columns of $A$.

In particular:

**Fact.** Testing whether a vector $b$ is in the span of some collection of vectors is equivalent to asking whether the augmented matrix with those columns is consistent.
When is Solution Guaranteed?

**Fact.** Matrix equation $Ax = b$ has a solution for every vector $b$ if and only if the columns of $A$ span $\mathbb{R}^m$, if and only if $A$ has a pivot in each row.

Proof uses the fact that if there is a pivot in each row, then the condition for inconsistent system cannot be satisfied. And if there is not a pivot in each row, then we can choose $b$ where the condition for inconsistent system holds.
Homogenous Systems

**Defn.** A *homogeneous system* is $Ax = 0$. It always has at least the *trivial* solution $x = 0$. 
The solution to a general linear system can be written in \textit{parametric vector form} as: one vector plus an arbitrary linear combination of vectors satisfying the corresponding homogeneous system.
An Example of Parametric Vector Form

Earlier we gave a solution as something like

\[ x_1 = -1 - 2x_2 + 3x_4 \]
\[ x_3 = 5 - x_4 \]

Now we add the equations \( x_2 = x_2 \) and \( x_4 = x_4 \). This system has solution:

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
\end{bmatrix} =
\begin{bmatrix}
  -1 \\
  0 \\
  5 \\
  0 \\
\end{bmatrix} + x_2 \begin{bmatrix}
  -2 \\
  1 \\
  0 \\
  0 \\
\end{bmatrix} + x_4 \begin{bmatrix}
  3 \\
  0 \\
  -1 \\
  1 \\
\end{bmatrix}
\]
Defn. A collection of vectors is **linearly independent** if the only linear combination of them that equals 0 is the trivial combination (all weights zero). Otherwise it is said to be **linearly dependent**.

Note that the collection is linearly dependent if some vector in it can be written as a linear combination of the other vectors.
Key Examples

œ Two vectors $u$ and $v$ are linearly dependent if and only if one is a multiple of the other. If they are linearly independent, then they span a plane through the origin. Further, inserting $w$ into the collection produces a linearly independent set if and only if $w$ is not in $Span\{u, v\}$.

œ A set containing the zero vector is automatically linearly dependent.
When Homogenrous System has Unique Solution?

**Fact.** The columns of matrix $A$ are linearly independent

$\iff Ax = 0$ has only the trivial solution

$\iff$ there is no free variable.
The matrix equation $Ax = b$ has a solution if and only if $b$ is a linear combination of the columns of $A$. This is guaranteed precisely when the columns of $A$ span $\mathbb{R}^m$; equivalently $A$ has a pivot in each row.

A homogeneous system equals 0. Parametric vector form represents the solution as: one vector plus arbitrary linear combination of vectors satisfying the homogeneous system.
A collection of vectors is linearly independent if the only linear combination of them that equals 0 is the trivial combination.

The homogenous system $Ax = 0$ has a unique solution precisely when the columns of $A$ are linearly independent; equivalently there is no free variable.