Vectors and Matrices
Vectors

**Defn.** *A matrix with one column is called a (column) *vector*.*

We use bold letters for vector variables, such as $x$ and $v$.

We sometimes write the column vector $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ as $(3, 5)$. 
Vector **addition** is performed by adding the corresponding entries. **Scalar multiplication** is performed by scaling each entry. That is,

\[
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} + \begin{bmatrix}
  v_1 \\
  v_2
\end{bmatrix} = \begin{bmatrix}
  u_1 + v_1 \\
  u_2 + v_2
\end{bmatrix}
\]

and

\[
c \begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} = \begin{bmatrix}
  cu_1 \\
  cu_2
\end{bmatrix}
\]

For example

\[
x \begin{bmatrix}
  2 \\
  4
\end{bmatrix} + y \begin{bmatrix}
  -1 \\
  7
\end{bmatrix} = \begin{bmatrix}
  2x - y \\
  4x + 7y
\end{bmatrix}
\]
**Defn.** We use $\mathbb{R}^d$ for the set of all $d$-entry vectors whose entries are real numbers.

One can associate vector in $\mathbb{R}^d$ with the corresponding point. For example, $\mathbb{R}^2$ is the 2-dimensional plane. And vector addition can be illustrated with a parallelogram:
Defn. A **linear combination** of vectors is formed by summing some multiple of each vector. The multipliers are called the **weights**.
**Defn.** The **span** of a collection of vectors is the set of all possible linear combinations. If $S$ is a set, we will denote its span by $\text{Span } S$.

For example, the span of a single (nonzero) vector is a line.

The span of two vectors is (usually) a plane.
**Defn.** If $A$ is an $m \times n$ matrix and $x$ is in $\mathbb{R}^n$, then the **matrix-vector product** $Ax$ is the linear combination of the columns of $A$ specified by $x$.

That is, if $A = [a_1, \ldots, a_n]$ (meaning its columns are vectors $a_1, \ldots, a_n$), and $x = (x_1, \ldots, x_n)$ then

$$Ax = x_1a_1 + x_2a_2 + \ldots + x_na_n$$
Example of Matrix-Vector Multiplication

For example,

\[
\begin{bmatrix}
2 & -1 \\
4 & 7
\end{bmatrix}
\begin{bmatrix}
3 \\
5
\end{bmatrix}
= 3 \begin{bmatrix}
2 \\
4
\end{bmatrix} + 5 \begin{bmatrix}
-1 \\
7
\end{bmatrix}
= \begin{bmatrix}
1 \\
47
\end{bmatrix}
\]
Summary

A vector is a matrix with one column. We use bold letters for vector variables. \( \mathbb{R}^d \) is all \( d \)-entry vectors with real entries. Vector addition adds corresponding entries; scalar multiplication scales each entry.

A linear combination of vectors is any sum of some multiple of each vector. Their span is the set of all possible linear combinations. The product of matrix \( A \) with vector \( x \) is the linear combination of columns of \( A \) given by \( x \).