Linear Equations, Back Substitution and Elementary Operations
A **linear equation** is that the sum of coefficients times variables is some value; for example, $3x - y = 7$. A **linear system** is a collection of linear equations.

A **solution** of the system satisfies all the equations. A system is **consistent** if it has a solution.
Example: two linear equations in the two variables $x$ and $y$. That is, two lines in the plane. Three possibilities: (a) intersect at unique point; (b) have nothing in common; (c) are the same line

**Fact.** There are exactly three possibilities for the solution set of a linear system: no solution, unique solution, or infinitely many solutions.
A linear system is **triangular** if the first equation has only one variable, the second equation only two variables, and so on.
**Triangular Systems Solved by Back-Substitution**

**ALGOR**  
**Back Substitution**  
- Solve the first equation for its variable.  
- Substitute the result into the second equation, and solve for its remaining variable.  
- Repeat.
Consider the system
\[
\begin{align*}
2x_1 &= 6 \\
x_1 + x_2 &= 2 \\
-x_1 + 4x_2 + x_3 &= 19
\end{align*}
\]
The first equation implies that \(x_1 = 3\). Substituting this into the second equation implies \(x_2 = -1\). Substituting both these values into the third equation implies \(x_3 = 26\). The solution is unique.
Matrices

**Defn.** A **matrix** is a rectangular arrangement of numbers. Its **size** is the number of rows and columns. We say “$m \times n$ matrix” to mean a matrix with $m$ rows and $n$ columns.
Augmented Matrix

A matrix can represent a linear system. The \textit{augmented} matrix is formed by adding the constants as the last column.

For example here is a system and its augmented matrix:

\[
\begin{align*}
2x_1 &= 6 \\
x_1 + x_2 &= 2 \\
-x_1 + 4x_2 + x_3 &= 19
\end{align*}
\]

\[
\begin{bmatrix}
2 & 0 & 0 & 6 \\
1 & 1 & 0 & 2 \\
-1 & 4 & 1 & 19
\end{bmatrix}
\]
Three Elementary Row Operations

- **Replacement**: Replace row by the sum of it and a multiple of another row; e.g. replace the second row by the sum of it and 3 times the first row. Often abbreviated to “add 3 times the first row to the second” or “$R_2' = R_2 + 3R_1$.”

- **Interchange**: Interchange two rows; e.g. swap the first and third rows.

- **Scaling**: Scale a row by nonzero factor; e.g. multiply entries in third row by 5.
Why interchange? System is unchanged.

Why scaling? System is unchanged.

Why replacement? Solution before remains solution. Since process reversible, cannot have introduced new solution.

**Defn.** Two matrices are row equivalent if can get from one to the other by elementary row operations.
Proofs

Every Fact in this course has a proof. We sometimes give the proof, sometimes sketch the highlights, and sometimes just skip it. Mathematics rests on proof. Proof provides a guarantee that the Fact is true. Proofs use logic, calculation, previous facts, and definitions.
Summary

A linear system is a collection of linear equations. A solution satisfies all the equations; the system is consistent if it has a solution.

A matrix is a rectangle of numbers in rows and columns. A linear system can be represented by an augmented matrix. Triangular systems can be solved by back-substitution.

The three elementary row operations are replacement, interchange, and scaling. Each preserves the solution set.