Calculating Determinants
Assume $A$ is an $n \times n$ matrix. Let $A_{ij}$ denote the matrix formed by removing row $i$ and column $j$. Then **expansion** across the first row of $A$ gives the formula

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{1+n} a_{1n} \det A_{1n}$$

One can expand across other rows, but note that the sign is always $(-1)^{i+j}$. Each term $C_{ij} = (-1)^{i+j} A_{ij}$ in the expansion is called a **cofactor**.
Example Cofactor Expansion

The matrix
\[
\begin{bmatrix}
3 & 6 & 0 \\
2 & 7 & -1 \\
0 & 4 & -8
\end{bmatrix}
\]
has determinant
\[
3 \begin{vmatrix} 7 & -1 \\ 4 & -8 \end{vmatrix} - 6 \begin{vmatrix} 2 & -1 \\ 0 & -8 \end{vmatrix} + 0 \begin{vmatrix} 2 & 7 \\ 0 & 4 \end{vmatrix}
\]
\[
= 3 \times (-52) - 6 \times (-16) + 0
\]
\[
= -60.
\]
\text{detTWO: 3}
Elementary Row Operations and Determinants

**Fact.**

- Adding a multiple of a row to another row does not change $\det$.
- Interchanging two rows flips the sign of $\det$.
- Multiplying a row by a scalar does the same to $\det$.
Calculating Determinants by Reduction

**ALGOR** If one obtains pivots in every row/column when reducing $A$ to echelon form, without using an interchange, then the determinant of $A$ is the product of the pivots.

$\text{detTW} = 5$
Example Calculation

The matrix
\[
\begin{bmatrix}
3 & 6 & 0 \\
2 & 7 & -1 \\
0 & 4 & -8
\end{bmatrix}
\]
reduces to
\[
\begin{bmatrix}
3 & 6 & 0 \\
0 & 3 & -1 \\
0 & 0 & -20/3
\end{bmatrix}
\]
without interchanges. Thus the determinant is
\[
3 \times 3 \times (-20/3) = -60.
\]
More Properties

\textbf{Fact.} \quad \det(A^T) = \det A \\
\det(AB) = (\det A)(\det B)
The **volume/area** of a box whose sides are vectors is given by the absolute value of the associated determinant.

For example, area of parallelogram determined by vectors \((x_1, y_1)\) and \((x_2, y_2)\) is \(|x_1y_2 - x_2y_1|\).

**Fact.** In \(\mathbb{R}^2\), if one applies a matrix transform \(M\) to some shape, then the area of the shape changes by a factor of \(\det M\).
Summary

A determinant can be calculated by cofactor expansion. Expansion across the first row says the determinant is the sum of the entry times the determinant of the matrix that remains when that row and column is deleted, with signs alternating.

Adding a multiple of a row to another row does not change $\det$, interchanging two rows flips the sign of $\det$, and multiplying a row by a scalar does the same to $\det$. 
If one obtains pivots in every row/column when reducing to echelon form, without using an interchange, then the determinant is the product of the pivots.

The determinant of the transpose is the same as that of the original. The determinant of the product is the product of the determinants. If one applies a matrix transform $M$ to some shape, then the area of the shape changes by a factor of $\det M$. 

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