15 Colorings and Planar Graphs

15.1 Bipartite Graphs

We saw already the complete bipartite graph. In general, we say that a graph is bipartite if one can partition the vertices into two sets, such that each edge has an end in each set. We saw earlier that trees are bipartite. For, one can pick any vertex as root and partition the vertices into two sets: the even generations and the odd generations. One can also think of the partition as a coloring of the vertices with two colors such that no edge joins two vertices of the same color.

An earlier exercise asked you to prove:

**Lemma 15.1** A cycle is bipartite if and only if it has an even number of vertices.

The connection between cycles and being bipartite is the following:

**Theorem 15.2** A connected graph is bipartite if and only if every cycle has even length.

**Proof.** There are two things to prove.

(1) If the graph is bipartite, then every subgraph must be bipartite. In particular, every cycle must have even length.

(2) Assume that every cycle has even length. Pick any vertex $v$. Then for $i \geq 0$, let $V_i$ be the set of vertices at distance $i$ from $v$. So, for example, $V_1$ is the neighbors of $v$, and $V_0 = \{v\}$. Then color red $v$ and every vertex at even distance from $v$; color the other vertices blue.

Now, we claim that the coloring is a bipartite partition. For, suppose there is an edge joining two vertices red vertices, say $a$ and $b$. By the definition of distance, $a$ and $b$ are in the same $V_i$. Then let $P_a$ and $P_b$ be a shortest path from $a$ and $b$ respectively back to $v$. This path contains one vertex from each $V_j$ for $j < i$. Now, the paths $P_a$ and $P_b$ meet up, at the latest at $v$. Let $w$ be the first vertex where they meet.
Then the segment \( a-w \) of \( P_a \) and the segment \( b-w \) of \( P_b \) have the same length. Thus, adding them and the edge from \( a \) to \( b \) produces an odd-length cycle, a contradiction. Hence, the coloring is a valid partition. ◊

It follows that there is a simple algorithm to test whether a connected graph is bipartite. Pick any vertex and color it red; color its neighbors blue; color their neighbors red; and so on. If one manages to color all vertices without a conflict, then the graph is bipartite; if one tries to color the same vertex with both colors, then the graph is not bipartite.

15.2 Colorings

A coloring of a graph means assigning colors to each vertex such that no edge joins two vertices of the same color. A \( k \)-coloring means a coloring that uses (at most) \( k \) colors. A graph having a 2-coloring is the same thing as being bipartite. The chromatic number of a graph, denoted \( \chi \), is the minimum number of colors needed for a coloring of the vertices.

Lemma 15.3 An even cycle has \( \chi = 2 \).
An odd cycle has \( \chi = 3 \).
The complete graph \( K_n \) has \( \chi = n \).

Proof. Left as an exercise. ◊

One application of chromatic number is the register allocation problem. In compiling a program, one would like to use the on-chip registers for as many of the variables as possible. So, construct a graph, where the vertices are the variables, and two vertices are connected by an edge if the corresponding variables can exist simultaneously. A \( k \)-coloring of the graph corresponds to an assignment of the variables to \( k \) registers.

Let \( \Delta \) be the maximum degree of a vertex in a graph. It is easy to show that the chromatic number is at most 1 more than the maximum degree:

Lemma 15.4 If \( G \) is a graph with chromatic number \( \chi \) and maximum degree \( \Delta \), then \( \chi \leq \Delta + 1 \).

Proof. Use a greedy algorithm. Color the vertices one at a time using the \( \Delta + 1 \) colors. Each time we color a vertex, we can pick any color not already present amongst its neighbors. That means there are at most \( \Delta \) forbidden colors. Hence the algorithm cannot get stuck, and creates a coloring with \( \Delta + 1 \) colors, as required. ◊

This result can be improved slightly by the following result, whose proof we omit:
Theorem 15.5 [Brooks] If $G$ is a connected graph that is not complete nor an odd cycle, then $\chi \leq \Delta$.

It is to be noted that, while testing for bipartiteness is easy, testing whether a graph has a 3-coloring appears to be much, much harder.

15.3 Planar Graphs

A **plane graph** is a graph drawn in the plane such that no pair of lines intersect. The graph divides the plane up into a number of regions called **faces**. Here are plane drawings of $K_4$ and $K_{2,3}$.

A **planar graph** is one which has a plane drawing. For example, every tree is planar.

Theorem 15.6 A connected plane graph has one face if and only if it is a tree.

This is actually much more difficult to prove rigorously than it looks.

Theorem 15.7 Euler's formula. For connected plane graph with $n$ vertices, $a$ edges, and $f$ faces:

\[ n - a + f = 2 \]

Proof. By induction on $a$. If $a = n - 1$, then $G$ is a tree and we’re done. Otherwise $a \geq n$. So there is a cycle containing some edge $e$. The removal of $e$ merges two faces. Let $G'$ be the resulting plane graph. Then $G'$ has $n$ vertices, $a - 1$ edges and $f - 1$ faces. And so, by the inductive assumption, $n - (a - 1) + (f - 1) = 2$. But the LHS is equal to $n - a + f$. 

It follows that:

Theorem 15.8 For any plane graph on $n$ vertices and $a$ edges, $a \leq 3n - 6$. 
Proof. Let $M$ be the number of edge–face pairs where the edge lies on the boundary of that face. Each edge appears twice: it lies on two boundaries. Each face appears at least three times. So we get $2a = M \geq 3f$. That is, $f \leq 2a/3$. Plug into Euler’s formula and do some algebra.  

Consequence: the complete graph $K_5$ is not planar.

A subdivision of a graph is created by adding some number of new vertices (possibly none) on each edge. A famous result is:

Theorem 15.9 Kuratowski’s Theorem. A graph is planar if and only if it does not contain a subdivision of either $K_5$ or $K_{3,3}$.

The most famous theorem in this area is the 4-Color Theorem. This was one of the first major theorems to make extensive use of a computer. It is due to Appel, Haken, and Koch.

Theorem 15.10 Four-Color Theorem. If $G$ is a planar graph, then $\chi(G) \leq 4$.

For you to do!

1. Draw a planar graph with 6 vertices and the maximum possible number of edges.

Exercises

15.1. The wheel $W_k$ is obtained from the cycle on $k$ vertices by adding one new vertex connected to all other vertices. Calculate the chromatic number of a wheel.

15.2. The prism of a graph $G$ on $n$ vertices is obtained by taking two separate copies of $G$ and adding $n$ “parallel” edges joining the corresponding vertices. For example, the prism of the hypercube $Q_n$ is $Q_{n+1}$. State and prove the relationship between the chromatic number of $G$ and the chromatic number of its prism.

15.3. Calculate the chromatic number of the following graph:
15.4. Let $T_m$ be graph constructed as follows. Start with $m^2$ nodes arranged in a square grid. Then join every node to the node above, below, to the right, and to the left, with wraparound. That is, each node in the top row is joined to the corresponding node in the bottom row (and vice versa); and each node in the leftmost column is joined to the corresponding node in the rightmost column (and vice versa). For example, $T_4$ is drawn here.

![Graph $T_4$]

Answer the following, with justification.

(a) When does $T_m$ have an Euler tour?
(b) When does $T_m$ have a Hamiltonian cycle?
(c) What is the chromatic number of $T_m$?

15.5. Show that the maximum number of edges in a bipartite planar graph with $n$ vertices is $2n - 4$.

15.6. Show that $K_{2,m}$ is planar for all $m$.

15.7. Show that the hypercube $Q_4$ is not planar.

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**Solutions to Practice Exercises**

1.