12 Sequences and Recurrences

A sequence is just what you think it is. It is often given by a formula known as a recurrence equation.

12.1 Arithmetic and Geometric Progressions

An arithmetic progression is a sequence where every two consecutive entries differ by the same amount. For example,

\[ 4, 7, 10, 13, 16, \ldots \]

If the sequence has first term \( A \) and last term \( L \) and there are \( n \) terms in the sequence, then an arithmetic progression has sum

\[ \frac{n(A + L)}{2} \]

(Proof left as exercise.)

A geometric progression is a sequence where every two consecutive entries have the same ratio. For example,

\[ 2, 6, 18, 54, 162, \ldots \]

A useful fact is the sum of a geometric progression:

\[ \sum_{i=0}^{n-1} Ar^i = \frac{A(r^n - 1)}{r - 1} \]

provided \( r \neq 1 \). This can be proved by induction, or by this idea:

\[ r \cdot \text{LHS} = \sum_{i=0}^{n-1} Ar^{i+1} = \sum_{i=1}^{n} Ar^i = \text{LHS} + Ar^n - A, \]

so that \( (r - 1)LHS = A(r^n - 1) \).

12.2 Fibonacci Numbers

Consider a board like a checkerboard that is partitioned into squares. Define a tiling of a board to mean covering the board completely with nonoverlapping dominoes, where each domino covers two adjacent squares. Consider a board with 2 rows and \( n \) columns. Clearly, there is a tiling with all vertical dominoes. But how many tilings are there? For example, if there are 3 columns, there are two other tilings, each with only 1 vertical domino:
Let \( f(n) \) be the number of domino tilings of the \( 2 \times n \) board. Let’s look for a way of writing \( f(n) \) in terms of smaller values. The idea is to look at the left end of the board. There are two possibilities: either there is a vertical domino at the left, or there are two horizontal dominoes. In the first case, the remaining dominoes form a tiling of the \( 2 \times (n - 1) \) board; in the second case, the remaining dominoes form a tiling of the \( 2 \times (n - 2) \) board. By the sum rule, we thus have:

\[
f(n) = f(n - 1) + f(n - 2).
\]

If there is only 1 column, then the recurrence formula breaks down, as two horizontal dominoes are impossible. If there are 2 columns, then the recurrence formula is valid, provided one defines \( f(0) = 1 \).

With some paper, one can calculate \( f(n) \), starting with \( f(0) \):

\[
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots
\]

These are the famous Fibonacci numbers. Gazillion things are counted by these.

They also have many patterns—here is one example.

**Theorem 12.1** For all \( n \geq 1 \),

\[
[f(n)]^2 - f(n - 1)f(n + 1) = (-1)^n.
\]

**Proof.** Proof by mathematical induction. The base case is \( n = 1 \). LHS = \([f(1)]^2 - f(0)f(2) = 1^2 - 1 \cdot 2 = -1\). RHS = \((-1)^1 = -1\). So the base case is true.

Now for the induction step. Assume the statement is true for \( n - 1 \); we need to prove it for \( n \). Well, start with the LHS for that case, use the definition of the Fibonacci sequence, and do some algebra:

\[
\begin{align*}
\text{LHS} &= [f(n)]^2 - f(n - 1)f(n + 1) \\
&= f(n)[f(n-1) + f(n-2)] - f(n-1)[f(n) + f(n-1)] \quad \text{(by Fibonacci defn twice)} \\
&= f(n)f(n-2) - f(n-1)f(n-1) \quad \text{(by simplification)} \\
&= -(-1)^{n-1} \quad \text{(by the induction hypothesis)} \\
&= (-1)^n = \text{RHS},
\end{align*}
\]

as required. \( \diamond \)
The original story behind the Fibonacci numbers was the following: A pair of rabbits starts breeding after two months and produces one pair every month thereafter. Assume we start with one new-born pair of rabbits. Let $R(n)$ be the number of pairs of rabbits after $n$ months. Then we claim that

$$R(n) = R(n - 1) + R(n - 2)$$

with $R(0) = R(1) = 1$. For, we still have the rabbits we had the month before. And every pair that is at least two months old produces a new pair. The number that are two months old are the ones that were alive two months ago.

There is a weird-looking formula for Fibonacci numbers:

$$f(n) = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right).$$

We shall see where this comes from in a moment.

### 12.3 Recurrence Equations

The formula for the Fibonacci numbers is an example of a recurrence. Here is another example.

**Example 12.1.** Find a recurrence for $S(n)$, the number of subsets of $\{1, 2, 3, \ldots, n\}$.

Every subset of $\{1, 2, \ldots, n-1\}$ can be extended to a subset of $\{1, 2, 3, \ldots, n\}$ by either adding or not adding the element $n$. Therefore

$$S(n) = 2S(n - 1) \text{ for } n \geq 1 \text{ and } S(0) = 1.$$ 

It follows immediately that $S(n) = 2^n$.

Most recurrence relations have **initial conditions**, since the recursive formula breaks down eventually for the smallest $n$. Note that without knowing the initial condition, the recurrence $S(n) = 2S(n - 1)$ has multiple solutions: $S(n) = \alpha 2^n$ is a solution for any real number $\alpha$ (including zero!). One can **verify** that some formula is a solution by plugging it into both sides and checking that one gets the same value. (Do it here!)

**Example 12.2.** Find a recurrence for $P(n)$, the number of unordered pairs from the set $\{1, 2, 3, \ldots, n\}$.
We saw already that \( P(n) = \binom{n}{2} \). But we can obtain a recurrence. Partition the pairs into two sets based on whether they contain the element \( n \) or not. There are \( n - 1 \) pairs that contain element \( n \), and \( P(n - 1) \) that don’t. So

\[
P(n) = n - 1 + P(n - 1)
\]

with initial condition \( P(1) = 0 \).

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**Example 12.3.** My money \( A \) is invested at interest rate of \( p \) percent compounding annually. What do I have after \( n \) years?

Let \( M(n) \) be the amount after \( n \) years. Then \( M(n) = (1+p/100)M(n-1) \), with \( M(0) = A \).

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### 12.4 Iterating the Recurrence

We can solve some recurrences by iterating them. This means repeatedly using the recurrence relation to re-write the RHS. (Actually, we can often get some information about them this way.)

For example: For our money from Example 12.3:

\[
M(n) = (1+p/100)M(n-1) = (1+p/100)^2M(n-2) = \ldots = (1+p/100)^nA.
\]

And for our pairs from Example 12.2:

\[
P(n) = (n - 1) + (n - 2) + \ldots + 1 + 0 = n(n - 1)/2
\]

where the last part uses the formula for the sum of an arithmetic progression.

Here is a harder example of solving a recurrence using iteration.

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**Example 12.4.** Solve

\[
T(n) = 4T(n-1) + 2^n \quad \text{with } T(0) = 6.
\]

Iterating the recurrence:

\[
T(n) = 2^n + 4T(n-1)
\]
\[
= 2^n + 4(2^{n-1} + 4T(n-2))
\]
\[
= 2^n + 2^n + 4^2T(n-2)
\]
\[
= 2^n + 2^n + 2^n + + 4^3T(n-3)
\]
\[
= \ldots
\]
\[
= (2^n + 2^n + \ldots + 2^{2n-1}) + 4^nT(0)
\]
\[
= (2^{2n} - 2^n) + 6 \cdot 4^n
\]
\[
= 7 \cdot 4^n - 2^n
\]
12.5 More Recurrences

A string of parentheses is called valid if it is of the correct form for an arithmetic expression. That is, the parentheses pair off such that every two pairs either nest or don’t overlap at all. For example, there are two valid strings of 4 parentheses: (( )) and ( ) ( ). (The valid strings have the property that, reading left to right, the number of left parentheses is always at least the number of right parentheses.)

Let \( p(n) \) be the number of valid strings using \( 2n \) parentheses. This has a slightly more interesting recurrence:

\[
p(n) = \sum_{i=1}^{n} p(i - 1) \times p(n - i) \quad (n \geq 1),
\]

with \( p(0) = 1 \).

To prove the recurrence. Consider any valid string of parentheses. Then the parentheses pair off. Consider the first parenthesis (a left one) and its partner. Suppose that it partners with the \( i^{th} \) right parenthesis. Then the string \( A \) between these two is itself a valid string of length \( 2(i - 1) \), and the string \( B \) after the partner is also a valid string of length \( 2(n - i - 1) \). Conversely, if you give me any two strings of valid parentheses of combined length \( 2(n - 1) \), I can recreate one of length \( 2(n) \) by \( (A)B \). It follows that the number of strings of valid parentheses where the first parenthesis pair off with the \( i^{th} \) right parenthesis equals \( p(i - 1) \times p(n - i) \). If we sum over all \( i \), we get the recurrence.

This solves to

\[
p(n) = \frac{1}{n + 1} \binom{2n}{n}
\]

While we don’t show how to find the solution, you can verify the solution yourself.

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**Example 12.5.** Find a recurrence for the number of domino tilings of a board with 3 rows and \( 2m \) columns.

Let \( t(m) \) be the number of such tilings. (Note that the number of columns has to be even for such a tiling to exist.) There are two possibilities for the first column.

1. All the dominoes are horizontal. Then they stretch over into the second column as well. What remains is a \( 3 \times (2m - 2) \) board. That is, the number of such possibilities is \( t(m - 1) \).

2. One domino is horizontal and one vertical. The horizontal domino is either the top or the bottom row. This gives a factor of 2. Say the horizontal domino is in the bottom row.
Then look at the top two rows. If the dominoes in the second column are horizontal, they force another horizontal domino in the bottom row. This keeps on going until eventually there is a vertical domino in the top two rows.

This can take 2, 4, 6 etc or even all the columns. Thus the number of possibilities for this case is

\[ 2 \left( t(m - 1) + t(m - 2) + t(m - 3) + \ldots + t(1) + t(0) \right) \]

where we have defined \( t(0) = 1 \). Thus one gets the recurrence

\[ t(m) = 3t(m - 1) + 2t(m - 2) + 2t(m - 3) + \ldots + 2t(1) + 2t(0) \]

One can also use this to find a simpler-looking recurrence: take the equivalent recurrence for \( t(m + 1) \), that is

\[ t(m + 1) = 3t(m) + 2 \sum_{i=0}^{m-1} t(i) \]

Then we get

\[ t(m + 1) - t(m) = 3t(m) - t(m - 1) \]

which simplifies to

\[ t(m + 1) = 4t(m) - t(m - 1) \]

For example, we get 1, 3, 11, 41, 153, . . .

### 12.6 Solving Recurrence Relations with Characteristic Equations

The recurrence relation for the Fibonacci numbers is a second-order recurrence, meaning it involves the previous two values. It is also linear homogeneous, meaning that every term is a constant multiplied by a sequence value. In general, one can write this as:

\[ g(n) = ag(n - 1) + bg(n - 2) \]

Now, it turns out that \( g(n) = r^n \)—where \( r \) is some fixed real number—is a solution to this recurrence under certain circumstances. What are those circumstances? Well, a trivial case is \( r = 0 \); but let’s assume \( r \neq 0 \). One can plug this alleged solution into both sides and see what must happen. The LHS is \( r^n \). The RHS is \( ar^{n-1} + br^{n-2} \). If we divide through by \( r^{n-2} \) (legal since \( r \neq 0 \)), we get that \( r^2 = ar + b \). Put another way, we need \( r \) to be a root (that is, a solution) of the following equation:

\[ x^2 = ax + b \]

This is called the characteristic equation.
Theorem 12.2  If the characteristic equation \( x^2 = ax + b \) has two distinct real roots \( r_1 \) and \( r_2 \), then the solution of the recurrence relation \( g(n) = ag(n-1) + bg(n-2) \) (\( n \geq 2 \)) is given by
\[
g(n) = \alpha r_1^n + \beta r_2^n,
\]
where \( \alpha \) and \( \beta \) are real numbers.

Proof. We have just shown that each of \( g(n) = r_1^n \) and \( g(n) = r_2^n \) is a solution. The theorem claims that these two functions are, in the terms of linear algebra, a basis for the solution space: every other solution is a linear combination of these two, and every linear combination of these two is indeed a solution. We omit the proof, but you should do it if you need the exercise.  

Example 12.6. Solve the recurrence \( R(n) = 5R(n-1) - 6R(n-2) \).

Method: the above theorem applies. The characteristic equation is \( x^2 = 5x - 6 \). We first solve the quadratic: the roots are \( r_1 = 2 \), \( r_2 = 3 \). So the general formula is \( R(n) = \alpha 2^n + \beta 3^n \) for some constants \( \alpha \) and \( \beta \). The constants \( \alpha \) and \( \beta \) can be obtained by looking at the initial conditions, which are the first two values of the sequence. We get two equations in two unknowns, which we then solve.

Let’s determine the solution for the Fibonacci numbers. The characteristic equation is \( x^2 = x + 1 \). By the quadratic formula, the roots of this are \( r_1 = (1 + \sqrt{5})/2 \) and \( r_2 = (1 - \sqrt{5})/2 \). So the solution is \( f(n) = \alpha r_1^n + \beta r_2^n \).

The coefficients \( \alpha \) and \( \beta \) are found by using the initial conditions, that is, that \( f(0) = f(1) = 1 \). In particular, we need that
\[
f(0) = \alpha + \beta = 1 \quad \text{and} \quad f(1) = \alpha r_1 + \beta r_2 = 1.
\]
And we get from algebra, that \( \alpha = r_1/\sqrt{5} \) and \( \beta = -r_2/\sqrt{5} \). This means that
\[
f(n) = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right).
\]
But things are actually simpler than they look: as \( n \to \infty \) the second term tends to zero, so actually
\[
f(n) \approx \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n.
\]
(Indeed, it can be shown that \( f(n) \) is the nearest integer to this quantity.)
One can also establish a similar result to the above theorem for the case where the two roots are not real, or for the case where there is a repeated root. The ideas for the latter are discussed in the exercises.

**Exercises**

12.1. Prove that if an \( n \)-term arithmetic progression has first term \( A \) and last-term \( L \), then its sum is \( n(A + L)/2 \).

12.2. Consider a board like a checkerboard that is partitioned into squares. Define a tromino tiling of a board to mean covering the board completely with nonoverlapping trominoes, where each tromino covers three squares in a row (horizontally or vertically). Let \( t(n) \) be the number of tromino tilings of the \( 3 \times n \) board. Give a recurrence formula for \( t(n) \).

12.3. In a rabbit warren, each pair of rabbits aged two months or more produces 2 pairs per month (and never dies).
   
   (a) Give a recurrence for the number of pairs after \( n \) months.
   
   (b) If we start with 1 newborn pair, how many rabbits do we have after one year?

12.4. Prove by induction that the Fibonacci number \( f(4m - 1) \) is a multiple of 3 for all \( m \geq 1 \).

12.5. Let \( S(n) \) be the number of strings of length \( n \) consisting of 0s and 1s such that no two 1s are consecutive. Determine a recurrence formula for \( S(n) \).

12.6. Prove that the Fibonacci sequence obeys the following identity:

\[
f(0) + f(1) + \ldots + f(n) = f(n + 2) - 1.
\]

12.7. Prove that the Fibonacci numbers obey the following identity:

\[
\sum_{i=0}^{n} [f(i)]^2 = f(n)f(n + 1) \quad \text{for } n \geq 0.
\]

12.8. Prove that every positive integer can be written as a sum of some distinct Fibonacci numbers with the added restriction that no two of the Fibonacci numbers used are consecutive. For example, \( 28 = f(7) + f(4) + f(2) \).

12.9. (a) Show that any two consecutive Fibonacci numbers \( f(n - 1) \) and \( f(n) \) are relatively prime.
(b) Code up or apply the Extended Euclid algorithm to find $s$ and $t$ such that $sf(n) + tf(n - 1) = 1$. Discuss your results, conjecture a pattern, and try to prove your conjecture.

12.10. Verify that the formula for $p(n)$, the number of valid strings of $2n$ parentheses, is correct by showing that it satisfies the recurrence.

12.11. Wayne has a membership at the FoolTiltPoker website that allows him to play any days that he likes, except that he can never play on three consecutive days. Let $p(n)$ be the number of possible subsets of days in an $n$-day period when Wayne can play.

(a) Explain why $p(3) = 7$.
(b) Calculate $p(4)$.
(c) Give a recurrence for $p(n)$.

12.12. Solve the recurrence $z(n) = 2z(n - 1) + 4^n$ with $z(0) = 1$ by iterating the recurrence.

12.13. Solve the recurrence $s(n) = 3s(n - 1) + 1$ with $s(0) = 1$ by iterating the recurrence.

12.14. The Lucas numbers $l(n)$ are like the Fibonacci numbers, except that they start differently: 2, 1, 3, 4, 7, 11, 18 . . . State and prove a formula for the sum of the first $n$ Lucas numbers.

12.15. Determine a formula for the Lucas numbers $l(n)$.

12.16. Give the general solution of the recurrence $g(n) = 2g(n - 1) + 3g(n - 2)$ by using the characteristic equation.

12.17. Solve the recurrence $h(n) = 6h(n - 1) - 8h(n - 2)$, with $h(1) = 4$ and $h(2) = 16$, by using the characteristic equation.

12.18. Consider the recurrence $h(n) = 4h(n - 1) + 4h(n - 2)$ ($n \geq 2$).

(a) Show that the characteristic equation has only one root.
(b) Show that both $h(n) = 2^n$ and $h(n) = n2^n$ are solutions to the recurrence.
(c) Suppose $h(0) = 1$ and $h(1) = 2$. Solve the recurrence.