13 More Graphs: Euler Tours and Hamilton Cycles

13.1 Degrees

The degree of a vertex is the number of edges coming out of it. The following is sometimes called the “First Theorem of Graph Theory”:

**Lemma 13.1** Suppose the graph has \( n \) vertices and \( a \) edges. Suppose the degrees of the graph are \( d_1, \ldots, d_n \). Then

\[
\sum_{i=1}^{n} d_i = 2a.
\]

**Proof.** This is a double-counting argument: the LHS and RHS count the same quantity, namely ends of edges. When we sum up the degrees of a graph, we are counting the ends of edges. Each edge has two ends, and so is counted twice. ◊

As a consequence we get:

**Lemma 13.2** In any graph, the number of vertices of odd degree is even.

**Proof.** We know from the previous result that the sum of the degrees is even. That means that there must be an even number of odd summands. ◊

13.2 A Few Good Graphs

The complete graph \( K_n \) on \( n \) vertices is the graph where every pair of vertices are joined by an edge. Thus \( K_n \) has \( \binom{n}{2} \) edges. The complete bipartite graph \( K_{r,s} \), for positive integers \( r \) and \( s \), has \( r + s \) vertices, split into two groups: \( r \) vertices on one side, and \( s \) vertices on the other. All edges go between the two sides. Here are \( K_4 \) and \( K_{3,3} \).
The hypercube is a more interesting graph. The hypercube $Q_q$ of dimension $q$ has $2^q$ vertices. Each vertex corresponds to a bit string (meaning 0’s and 1’s) of length $q$. Two vertices are joined by an edge if their corresponding strings differ in exactly one bit. Each vertex therefore has degree $q$. We can also define $Q_q$ recursively: To form $Q_q$, take two copies of $Q_{q-1}$ and join each pair of corresponding vertices by an edge. Here is $Q_3$.

A subgraph of a graph $G$ is a graph that contains some of the edges and some of the vertices of the graph $G$. A subgraph is a spanning subgraph if it contains all the vertices of the original graph.

### 13.3 Eulerian Graphs

For a famous example of a problem, consider the problem of drawing the following picture without lifting your pen and without going over the same line more than once. Try it!

Did you manage to do this? There is a slight trick in that one must start at the bottom of the picture. Why?

A tour is a walk in a graph that does not use any edge more than once and ends up where it started. An Euler tour is a walk that goes along every edge exactly once, and ends up where one started. This is like the continuous pen drawing, except with the added requirement that one ends at the same place one begins.

**Theorem 13.3** A connected graph has an Euler tour if and only if every vertex has even degree.
Proof. There are two things to prove. We will prove that if the graph has an Euler tour, then every vertex has even degree. And if every vertex has even degree, then the graph has an Euler tour.

(1) Assume we have an Euler tour. If a vertex has odd degree, then it must be the start or finish of the tour, since if we don’t start there, then every time we pass through the vertex we use two of its edges, and so the last time we arrive we are stuck. A graph cannot have exactly one odd vertex (by Lemma 13.2); so it must be that we have no odd vertex.

(2) The second part takes a little bit more work. Define a tour decomposition as a collection of tours that use up all the edges of the graph. Assume the graph has every vertex of even degree. Then we claim there is a tour decomposition.

One can construct a tour decomposition in a simple blind fashion: start tracing out a tour until one gets stuck. Since a tour uses an even degree at each vertex, once we remove the first tour, what remains must have even degree throughout. Thus, one can repeat the process, thereby using up all the edges.

But what we want is an Euler tour. This is a tour decomposition consisting of only one tour. So how about this: take the tour decomposition with the minimum number of tours. And suppose there are at least two tours in this decomposition.

Then, because the original graph is connected, there must be somewhere two tours that share a vertex, call it $p$. We can re-organize things such that $p$ is the start and finish of both tours. But then it is easy to merge the two tours, by continuing on the second after finishing the first. Hence we get a tour decomposition with fewer tours, a contradiction. What was the problem? We supposed that there were two or more tours. In fact, we have an Euler tour. ♦

The above is a constructive proof: it provides an algorithm for finding an Euler tour. A somewhat more efficient algorithm is to construct the tour directly: start at any vertex, trace along any unused edge and keep on going, except that if there is a bridge-edge at a vertex (meaning an edge whose removal would increase the number of components in the remaining graph), you must take it. This algorithm is sometimes attributed to Fleury.

13.4 Hamiltonian Graphs

A Hamilton cycle is a cycle that visits every vertex exactly once. That is, a Hamilton cycle is a spanning cycle. Similarly, a Hamilton path is a path that visits every vertex exactly once. This idea sounds similar to Euler, but not really. No simple characterization of when a graph has a Hamilton cycle is known. Indeed, it is strongly believed that such a characterization does not exist.
The problem of determining whether a graph has a Hamilton cycle has been proven to be **NP-complete**: while we do not define this concept here, we do point out that there is a $1$ million prize offered for a proof or disproof of the conjecture that none of the NP-complete problems has a polynomial-time algorithm. That is, if you can find an algorithm that runs in at most time proportional to $n^{1000000}$ for all graphs and is **guaranteed** to determine whether the graph has a Hamilton cycle, then you’re a millionaire. Note that there are around $n!$ possible cycles, so checking them all is not going to be anywhere near fast enough.

**Lemma 13.4**  
(a) The complete graph $K_n$ has a Hamilton cycle for $n \geq 3$.  
(b) The complete bipartite graph $K_{r,s}$ has a Hamilton cycle if and only if $r = s \geq 2$.

We leave the proof of part (b) as an exercise.

The hypercube has a Hamilton cycle. Indeed, Hamilton cycles in the hypercube are called **Gray codes**, and are important in communication. For example, in $Q_3$ one Gray code is $000, 001, 011, 010, 110, 111, 101, 100, 000$.

We content ourselves with one sufficient condition and one necessary condition.

**Theorem 13.5** Let $G$ be a graph with $n$ vertices. If every vertex has degree at least $n/2$, then $G$ has a Hamilton cycle.

**Proof.** Suppose the theorem is false. That is, there exists a counterexample on $n$ vertices for some $n$. Then let $G$ be the counterexample on $n$ vertices with the **maximum** number of edges.

Let $v_1$ and $v_n$ be some pair of vertices not joined by an edge. Since $G$ is a maximum counterexample, it follows that if we add the edge $v_1v_n$ to $G$, then we have a Hamilton cycle. That is, $G$ has a Hamilton path, say $v_1, v_2, v_3, \ldots, v_n$.

Now, by the hypothesis of the theorem, there are $n$ edges from $v_1$ and $v_n$ combined to the rest of the graph. This means that, apart from the edges $v_1v_2$ and $v_{n-1}v_n$, there are $n-2$ edges coming out of $v_1$ and $v_n$. For $2 \leq i \leq n-2$, let $P_i$ be the pair of potential edges $v_1v_{i+1}$ and $v_nv_i$. It follows that there must be an $i$ such that both edges in $P_i$ exist; that is, $v_1$ is adjacent to $v_{i+1}$ and $v_n$ is adjacent to $v_i$.

![Diagram](image)

By adding in these two edges and deleting the edge $v_iv_{i+1}$, this gives us a Hamilton cycle, a contradiction.  

\[\Diamond\]
Theorem 13.6 If graph $G$ has a Hamilton cycle, then for every set $S$ of vertices, the number of components of $G - S$ is at most $|S|$.

Proof. Consider the Hamilton cycle $C$ of $G$. Mark the vertices of $S$ on $C$. Between each vertex of $S$ we have a piece of $C$. There are at most $|S|$ pieces of $C - S$, and thus at most $|S|$ components of $G - S$. $\Diamond$

The converse of this theorem is false.

Exercises

13.1. Draw all connected graphs with 5 vertices and 5 edges.

13.2. Draw all (simple) graphs with 5 vertices and 7 edges.

13.3. The degree sequence of a graph is the sorted sequence of its degrees. Draw a graph with the degree sequence 4, 3, 2, 2, 1, 0.

13.4. In both the following cases, draw a tree with that degree sequence or prove that it is impossible:
   
   (a) 4, 3, 3, 2, 1, 1, 1, 1, 1
   
   (b) 4, 3, 3, 2, 1, 1, 1, 1, 1

13.5. Characterize the degree sequences of trees. That is, state and prove a theorem of the form: There is a tree with degree sequence $d_1, d_2, \ldots, d_n$ if and only if . . .

13.6. Assuming vertices are indistinguishable, draw all (unrooted) trees that have exactly 7 vertices of which exactly 2 vertices have degree exactly 3.

13.7. A happy tree is a tree where every vertex has degree 1 or 3.

   (a) Draw a happy tree with 10 vertices.
   
   (b) How many different happy trees are there with 10 vertices, assuming vertices are indistinguishable?
   
   (c) Prove that there is no happy tree with 11 vertices.

13.8. Let $T$ be some fixed tree with 101 vertices.

   (a) Show that if every vertex of a graph $G$ has degree at least 100, then we can find a copy of $T$ as a subgraph of $G$. 
(b) Show that the conclusion does not necessarily hold if every vertex of $G$ has degree 99.

13.9. Does the following graph have a Hamilton path? A Hamilton cycle?

![Graph](image)


13.11. How many different Hamilton cycles does $K_n$ have? (Note that the answer depends on exactly what one means by two cycles being different; so explain your choice.)

13.12. The Wheel graph $W_n$ is obtained by taking a cycle with $n$ vertices and adding one new vertex that is joined to every vertex in the cycle. Complete the following, with justifications:

(a) $W_n$ has an Euler tour if and only if
(b) $W_n$ has a Hamilton cycle if and only if

13.13. Define the graph $G_m$ as the graph that is a grid of $3m$ vertices arranged in 3 rows and $m$ columns such that each vertex has an edge to the vertices to the left, above, to the right, and below it, if they exist. For example, $G_4$ is illustrated here.

![Graph](image)

Complete the following, with justifications:

(a) $G_m$ has an Euler tour if and only if
(b) $G_m$ has a Hamilton cycle if and only if

13.14. A tournament is obtained by taking the complete graph $K_n$ and orienting every edge to form a directed graph (where every road is a one-way street).

(a) Show that a tournament always has a directed Hamilton path.
(b) Show that a tournament might not have a directed Hamilton cycle.

13.15. Using the Internet if necessary, look up the term “change ringing”. Explain the connection with Hamilton cycles.

13.16. Using the Internet if necessary, discuss one method to produce Gray codes.