WORM colorings Forbidding Cycles or Cliques

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Abstract. Given a coloring of the vertices, we say subgraph \( H \) is monochromatic if every vertex of \( H \) is assigned the same color, and rainbow if no pair of vertices of \( H \) are assigned the same color. Given a graph \( G \) and a forbidden graph \( F \), we define an \( F \)-WORM coloring of \( G \) as a coloring of the vertices if \( G \) without a rainbow or monochromatic subgraph isomorphic to \( F \). We explore such colorings especially as regards to the existence, complexity, and optimization within certain graph classes. Our focus is on the case that \( F \) is a cycle or a complete graph.

1 Introduction

Let \( F \) be a graph. Consider a coloring of the vertices of graph \( G \). We say that a copy of \( F \) (as a subgraph) is rainbow if all its vertices have different colors. We say that a copy of \( F \) (as a subgraph) is monochromatic if all its vertices receive the same color. Then an \( F \)-WORM coloring of \( G \) is a coloring of the vertices of \( G \) Without a Rainbow or Monochromatic subgraph isomorphic to \( F \). For example, if \( G = K_5 \), then any coloring of \( G \) yields either a monochromatic \( K_3 \) or a rainbow \( K_3 \). On the other hand, if we color \( G = K_4 \) giving two vertices red and two vertices blue, we avoid both a rainbow and a monochromatic \( K_3 \).

Further, if the graph \( G \) has an \( F \)-WORM coloring, then we define \( W^+(G, F) \) as the maximum number of colors and \( W^-(G, F) \) as the minimum number of colors in such a coloring.

This concept for vertex colorings was introduced in [8]. However, the idea of forbidding both a monochromatic subgraph and a rainbow subgraph has been extensively studied for edge-colorings; see, for example, Axenovich and Iverson [1] and other work on “anti-Ramsey” numbers. Similarly, there is, of course, much work on avoiding monochromatic subgraphs, such as [4, 10, 12]. There is also some research on avoiding rainbow subgraphs. For example, Bujtás et al. [3] defined the star-[\( k \)] upper chromatic number as the maximum number of colors in a coloring of the vertices without a rainbow \( K_{1,k} \).
In [8] the focus was on the case that $F$ is a path or a star. In this paper we focus on the case that the forbidden subgraph is a cycle or a clique. Specifically, in Section 2 we examine when graphs have a $K_3$-WORM coloring, showing that the question is NP-complete for general graphs and calculating or bounding $W^+(G, K_3)$ and $W^-(G, K_3)$ for some families of graphs $G$. Section 3 considers similar questions where $C_4$ and other cycles are forbidden, while in Section 4 a clique or biclique is forbidden. In Section 5 we introduce the related question of coloring a graph to minimize the total number of monochromatic and rainbow subgraphs.

2 Forbidding a Triangle

We consider here the case that the forbidden graph $F$ is the triangle. As noted above, the largest complete graph that has a $K_3$-WORM coloring is $K_4$. Of course, if graph $G$ does not have a triangle, then one can color every vertex the same color or every vertex a different color and have a WORM coloring. Thus the interesting graphs are those with clique numbers 3 and 4. We focus our attention on two classes of graphs: (maximal) outerplanar graphs and cubic graphs. We also show that, as expected, determining whether a graph has a $K_3$-WORM coloring is NP-complete.

2.1 Outerplanar graphs

Any outerplanar graph has a $K_3$-WORM coloring with 2 colors. This follows from the fact that every outerplanar graph has (vertex) arboricity 2. That is, one can 2-color the vertices without any monochromatic cycle. Thus the main question is about $W^+$ in outerplanar graphs.

Note that the value of $W^+(G, K_3)$ is not the same as the maximum number of colors without a rainbow $K_3$. For example, consider the Hajós graph $H$. Then $W^+(G, H) = 3$ (see below), but one can use four colors to avoid a rainbow $K_3$ by coloring all the vertices of the central triangle the same color. We focus on maximal outerplanar graphs (MOPs).

**Theorem 1** For a MOP $G$, it holds that $W^+(G, K_3) = m(D(G)) + 2$ where $D(G)$ is the weak dual of $G$ and $m(H)$ is the matching number of graph $H$.

**Proof.** Consider a MOP $G$ and a $k$-coloring of its vertices. Let $F$ be the set of monochromatic edges. Note that the coloring is a WORM-coloring if and only if every triangle of $G$ has precisely one edge in $F$. Let $H$ be the spanning subgraph with edge set $F$.
Since $G$ is chordal, if $H$ contains a cycle then it contains a triangle. So $H$ is a forest. Furthermore, since $k$ is at most the number of components of $H$, we have $k \leq n - h$, where $n$ is the number of vertices in $G$ and $h$ is the number of edges in $H$.

Now, let $M$ be the edges of $H$ that are chords in $G$. Each edge in $M$ corresponds to an edge in the weak dual $D(G)$. Since $H$ contains exactly one edge from each triangle, this means that the edges of $M$ correspond to a matching $M'$ in $D(G)$. The edges of $H$ not in $M$ correspond to isolated vertices in $D(G)$. Since $D(G)$ has $n - 2$ vertices, we have $h = n - 2 - |M'|$. And thus $k \leq n - ((n - 2) - |M'|) = |M'| + 2 \leq m(D(G)) + 2$. This proves the upper bound.

Next we prove the lower bound. Note that $D(G)$ is a tree. It is known that every tree with at least one edge has a maximum matching that saturates all its non-leaf vertices (see, for example, [9]). So consider a maximum matching $M'$ of $D(G)$ that saturates all its non-leaf vertices. That $M'$ corresponds to a set $M$ of chords of $G$. Each vertex of $D(G)$ not incident with $M'$ correspond to a triangle in $G$ that has two sides on the outer face. So we can for each such triangle choose one of the edges on the outer face, and add to $M$ to form a set of edges $F$ in $G$ that contain exactly one edge from each triangle.

So, take the spanning subgraph with edge-set $F$ and give every component a different color. Thus we have a WORM-coloring with, by the same arithmetic as above, $|M'| + 2$ colors.

It follows that the maximum value of $W^+(G, K_3)$, taken over all MOPs $G$ of order $n$, is $n/2 + 1$. This is attained by multiple graphs, including the fan (obtained from a path by adding one vertex adjacent to every vertex on the path) and the MOPs of maximum diameter (obtained from an $n$-cycle by first adding a noncrossing matching of $n/2 - 1$ chords and then adding more chords to make the result into a MOP). The MOPs with the minimum value of $W^+(G, K_3)$ for their order are those whose duals are the trees of maximum degree 3 that have minimum matching number.

### 2.2 Cubic graphs

We now consider 3-regular graphs. Lovász [10] showed that a cubic graph $G$ has a 2-coloring where every vertex has at most one neighbor of the same color. Thus, $W^-(G, K_3)$ is 2 if $G$ has a triangle, and 1 otherwise.

So, as in outerplanar graphs, the focus is on $W^+(G, K_3)$. Although we do not have a general formula, we can determine the extremal values. The maximum value for a given order is trivial: $W^+(G, K_3)$ is the order of $G$. 


if the graph is triangle-free. The following result determines the minimum value for a given order:

**Theorem 2** For any cubic graph $G$ on $n$ vertices with $n \geq 6$, $W^+(G, K_3) \geq \frac{2n}{3}$.

**Proof.** Let $H$ be the spanning subgraph of $G$ whose edges are those edges of $G$ that lie in a triangle. Then $H$ is a union of disjoint copies of $K_3$ and $K_4 - e$ (and isolated vertices). It follows that there is a matching $M$ of cardinality at most $n/3$ whose removal destroys all triangles in $G$. For each edge $e$ in $M$, create one color $c_e$ and color both ends of $e$ with color $c_e$. Give every other vertex in $G$ a distinct color. This gives a $K_3$-WORM coloring that uses at least $\frac{2n}{3}$ colors. $\text{QED}$

There is equality in the above bound if and only if the graph $G$ has a 2-factor consisting of triangles.

### 2.3 Cartesian products

Since a $K_3$ in a Cartesian product has to lie completely within one of the fibers, it is not surprising that the product $G \square H$ has a $K_3$-WORM coloring if and only if both graphs $G$ and $H$ have $K_3$-WORM colorings. Indeed, a coloring of the product $G \square H$ can be produced in the standard way of combining colors. Thus:

**Observation 1** Assume graphs $G$ and $H$ have $K_3$-WORM colorings. Then

$W^-(G \square H, K_3) = \max\{W^-(G, K_3), W^-(H, K_3)\}$ and $W^+(G \square H, K_3) \geq W^+(G, K_3) \times W^+(H, K_3)$.

**Proof.** For the upper bound on $W^-(G \square H)$, take the minimum colorings of graphs $G$ and $H$ and think of them as integers in the range 1 to $W^-(G)$ and 1 to $W^-(H)$. Then color vertex $(g, h)$ of $G \square H$ by the sum of the colors of $g$ and $h$, arithmetic modulo $\max\{W^-(G), W^-(H)\}$.

For the lower bound on $W^+(G \square H)$, take the maximum colorings of graphs $G$ and $H$ and color vertex $(g, h)$ with the ordered pair of colors. $\text{QED}$

### 2.4 Complexity

We show next that determining whether a graph has a $K_3$-WORM coloring is hard.
We will need the following gadget. Let $G_7$ be the graph given by the join of $K_2$ and $C_5$. The graph $G_7$ has a $K_3$-WORM coloring: color the two dominating vertices red and the other five vertices blue. In fact, this coloring is unique:

**Observation 2** In any $K_3$-WORM coloring of $G_7$, the two dominating vertices receive the same color.

**Proof.** Suppose the two dominating vertices received different colors, say red and blue. Then by the lack of rainbow triangles, every vertex on the 5-cycle must be red or blue. But then there must be two consecutive vertices with the same color, which yields a monochromatic triangle, a contradiction. QED

**Theorem 3** Determine whether a graph $G$ has a $K_3$-WORM coloring is NP-complete.

**Proof.** We reduce from NAE-3SAT (not all equal 3SAT) (see [6]). Given a boolean formula in conjunctive normal form with three literals per clause, the NAE-3SAT problem is whether there is a truth assignment with at least one true literal and one false literal for each clause.

Given a boolean formula $\phi$, we build a graph $G_\phi$ as follows. Start with two master vertices $M_1$ and $M_2$ joined by an edge. For each variable $x$, create two vertices labeled $x$ and $\bar{x}$ joined by an edge and join both to both $M_1$ and $M_2$. Then, pick one variable, say $x_1$, and add a $C_5$ all vertices of which are joined to $x_1$ and to $M_1$. Now, for each clause $c$, create a triangle $T_c$ of three vertices. For each vertex of each clause triangle, join it to its corresponding literal and add a 5-cycle adjacent to both those vertices. An example is shown in the figure below.
Now, we claim that $G_\phi$ has a $K_3$-WORM coloring if and only if $\phi$ has an NAE assignment. So assume that $G_\phi$ has such a coloring. The four vertices $x_1, \bar{x}_1, M_1,$ and $M_2$ form a $K_4$. So two of these vertices are one color, say red, and two are another color, say blue. By Observation 2, $x_1$ and $M_1$ have the same color. So it must be that $M_1$ and $M_2$ have different colors. Further, for every variable $x$ we have that $\{x, \bar{x}, M_1, M_2\}$ forms a $K_4$. That is, for every pair $x$ and $\bar{x}$, exactly one is red and one is blue.

By Observation 2, it follows that the vertices in the clause triangle are colored by the same colors as the constituent literals. To avoid monochromatic triangles, it must be that not all the literals are equal. That is, we have an NAE assignment.

Conversely, given an NAE assignment for $\phi$, we can color $G_\phi$ with colors red and blue as follows. Color literal vertices red if they are true and blue if they are false. Color $M_1$ the same color as $x_1$ and $M_2$ the other color. Color the $C_5$’s the opposite color of their neighbors.

This shows that we have reduced NAE-3SAT to the $K_3$-WORM coloring problem, as required. QED

2.5 Conjecture on $W^-(G, K_3)$

In [8] we showed that if a graph has a $P_3$-WORM coloring, then it has such a coloring using 2 colors. That proof does not generalize. However, we conjecture that the result carries over:

**Conjecture 1** If a graph has a $K_3$-WORM coloring, then it has a $K_3$-WORM coloring using (at most) 2 colors.

Note that this conjecture does not extend to larger forbidden subgraphs. For example, any $K_4$-WORM coloring of $K_9$ uses three colors. However, this does not preclude an analogous result with “two” replaced by some constant that depends on the forbidden subgraph $F$.

Further, it remains unclear whether every graph $G$ has a coloring using $k$ colors for every $k$ between $W^-(G, F)$ and $W^+(G, F)$. Such a result is easily proved by induction if $G$ is a tree.

3 Forbidding a 4-Cycle or All Cycles

In this section we consider WORM colorings where a cycle is forbidden. Specifically, we consider $C_4$ as a forbidden subgraph, as well as extending the notion of WORM colorings to forbid all cycles.
For a set $F$ of graphs, we define an $F$-WORM coloring as one with no rainbow nor monochromatic copy of any graph in $F$. We define $W^-(G,F)$ and $W^+(G,F)$ as expected. Let $C$ denote the set of all cycles.

### 3.1 Outerplanar and Cubic Graphs

As we recalled earlier, every outerplanar graph $G$ has vertex arboricity at most 2. That partition shows that $G$ has a $C$-WORM coloring. The next result shows that $W^+(G,C)$ for such a graph is at least one more than the diameter of $G$.

**Theorem 4** For an outerplanar graph $G$, we have $W^+(G,C) \geq \text{diam}(G) + 1$.

**Proof.** Let $G$ be an outerplanar graph and $v_0$ be a vertex with maximum eccentricity. For $i \geq 0$, let $V_i$ be the set of vertices that are at distance $i$ from $v_0$. Define a coloring $c$ by giving color $i$ to vertices in $V_i$. Clearly this coloring uses $\text{diam}(G) + 1$ colors.

It is known that in an outerplanar graph the set of vertices at distance $i$ from a vertex $v_0$ form a linear forest. (A cycle in $V_i$ would yield a subdivision of $K_4$ in $G$; see, for example, [11].) So this coloring $c$ has no monochromatic cycle.

Further, a rainbow subgraph has at most one vertex from each $V_i$. Since there is no edge between $V_i$ and $V_j$ for $|i − j| ≥ 2$, it follows that all rainbow subgraphs are paths. In particular there is no rainbow cycle. \[\text{QED}\]

By Theorem 1, we have equality in the above theorem for MOPs of maximum diameter (that is, MOPs of diameter $\lfloor n/2 \rfloor$). However, the bound does not appear sharp if one forbids only a specific cycle, such as $C_4$. For example, computer search shows that $W^+(G,C_4) = 7$ for every MOP $G$ of order 9.

Similarly, since cubic graphs have vertex arboricity 2 (or by Lovász’ result), it follows that $W^+(G,C) \leq 2$ for a cubic graph $G$. Here is one calculation for $W^+(G,C)$:

**Observation 3** If $G$ is the Petersen graph, then $W^+(G,C) = 7$.

**Proof.** Suppose we have a $C$-WORM coloring with $k$ colors. Then if we keep one vertex of each color and discard the other vertices, what remains must be an acyclic graph. It follows that $n − k \geq \nabla(G)$, where $\nabla(G)$ denotes
the decycling number. (See [2] for a discussion of that parameter.) Since the decycling number of the Petersen graph is 3, we have that $W^+(G, C) \leq 7$.

For the lower bound, take a maximum independent set $I$ (which has size 4) and color all of its vertices red while giving all other vertices distinct colors. Since this is a proper coloring, there is no monochromatic cycle. To see that there is no rainbow cycle, one needs to check that every cycle in $G$ intersects $I$ in at least two vertices. QED

We consider next the question for cubic graphs forbidding only the 4-cycle. The maximum value of $W^+(G, C_4)$ is the order. We do not know the minimum value of $W^+(G, C_4)$ for cubic graphs $G$ of a given order, but a candidate is the following graph. Take $r$ disjoint copies of $K_{3,3} - e$ and add edges so that the resultant graph $B_r$ is cubic and connected. For example, $B_2$ is drawn in the figure below. The graph $B_r$ has $6r$ vertices and one can calculate that $W^+(B_r, C_4) = 4r$. Computer search shows that $B_2$ is the unique (connected) cubic graph $G$ of order 12 with minimum $W^+(G, C_4)$.

3.2 Cartesian Products

For the Cartesian product $G \Box H$ to have a $C_4$-WORM coloring, it is of course necessary that both $G$ and $H$ have a $C_4$-WORM coloring. However, unlike the case for forbidding $K_3$, we have not been able to show that this condition is sufficient, nor could we find an example to the contrary.

Recall that a Rooks graph is the Cartesian product of two cliques. We next consider the problem of avoiding monochromatic and rainbow cycles in a Rooks graph.

**Observation 4** The largest Rooks graph that has a WORM $C_4$-coloring is $K_9 \Box K_9$.

**Proof.** Note that any coloring of $K_{10}$ must contain either a monochromatic $C_4$ or a rainbow $C_4$. On the other hand, a $C_4$-WORM coloring of
$K_9 \square K_9$ is shown below.

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 \\
3 & 3 & 3 & 1 & 1 & 2 & 2 & 2 \\
2 & 2 & 2 & 3 & 3 & 3 & 1 & 1 \\
1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 \\
3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 \\
2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\
1 & 2 & 3 & 2 & 3 & 1 & 3 & 1 \\
3 & 1 & 2 & 1 & 2 & 3 & 2 & 3 \\
2 & 3 & 1 & 3 & 1 & 2 & 1 & 2 \\
\end{array}
\]

(Note that the above is equivalent to a partition of the edge-set of $K_{9,9}$ into three $C_4$-free cubic graphs. As such it was inspired by the construction used to prove the bipartite Ramsey number $b(2, 2, 2) \geq 11$ [5, 7].) QED

For grid graphs we believe:

**Conjecture 2** For $G$ the $m \times n$ grid,

\[
W^+(G, C_4) = \left\lfloor \frac{(m+1)(n+1)}{2} \right\rfloor - 1.
\]

This value is a lower bound, as we now show.

**Observation 5** For $G$ the $m \times n$ grid,

\[
W^+(G, C_4) \geq \left\lfloor \frac{(m+1)(n+1)}{2} \right\rfloor - 1.
\]

**PROOF.** We construct a suitable coloring. There are two cases.

Consider first the case that at least one of $m$ or $n$ is odd, say $n$. Think of the grid as having $n$ rows numbered 1 up to $n$. Then in each odd-numbered row, give every vertex a unique color. For even-numbered rows, give every vertex the same color. This coloring has no monochromatic cycle, no rainbow cycle, and uses $m(n+1)/2 + (n-1)/2 = (m+1)(n+1)/2 - 1$ colors.

Consider second the case that both $m$ and $n$ are even. Still think of the grid as having $n$ rows numbered 1 up to $n$. Then color all but the last two rows as before: in each odd-numbered row give every vertex a unique color, and in even-numbered rows give every vertex the same color. For the last two rows, do the same coloring but based on columns: that is, in each odd-numbered column give both vertices a unique color, and in even-numbered
columns give both vertices the same color. The coloring for \( m = n = 6 \) with 23 colors is illustrated below, where • means that the vertex receives a distinct color.

\[
\begin{array}{cccccc}
• & • & • & • & • & • \\
1 & 1 & 1 & 1 & 1 & 1 \\
• & • & • & • & • & • \\
2 & 2 & 2 & 2 & 2 & 2 \\
• & 3 & • & 4 & • & 5 \\
• & 3 & • & 4 & • & 5 \\
\end{array}
\]

This coloring has no monochromatic cycle, no rainbow cycle, and uses \( m(n-2)/2 + (n-2)/2 + 3(m/2) = (m+1)(n+1)/2 - 3/2 \) colors. \[\text{QED}\]

4  Forbidding a Clique or Biclique

We consider next the case that the forbidden subgraph is a complete graph.

**Theorem 5** (a) \( K_n \) has a \( K_m \)-WORM coloring if and only if \( n \leq (m-1)^2 \).

(b) In this range, \( W^+(K_n, K_m) = m - 1 \) and \( W^-(K_n, K_m) = \lceil n/(m-1) \rceil \).

**Proof.**  (a) Assume \( K_n \) has a \( K_m \)-WORM coloring. Let \( r \) be the number of colors; since there is no rainbow \( K_m \), \( r \leq m - 1 \). Since there is no monochromatic \( K_m \), each color class contains at most \( m - 1 \) vertices. Hence \( n \leq (m - 1)^2 \). On the other hand, if \( n \leq (m - 1)^2 \), then we use \( m - 1 \) colors as equitably as possible. This coloring has no monochromatic or rainbow \( K_m \).

(b) Let \( r \) be the number of colors in a \( K_m \)-WORM coloring of \( K_n \). If \( r \geq m \), then there is a rainbow \( K_m \). So \( W^+ \leq m - 1 \). Further, each color is used at most \( m - 1 \) times. Therefore, \( r \geq n/(m-1) \).

(We note that we can have \( r \) as any value between \( n/(m-1) \) and \( m - 1 \): just use each of the \( r \) colors as equitably as possible.) \[\text{QED}\]

Part (a) of the above theorem can be extended to say that: if graph \( G \) has chromatic number at most \( (m-1)^2 \), then it has a \( K_m \)-WORM coloring. One colors \( G \) with \( m - 1 \) colors, giving all the vertices in \( m - 1 \) color classes the same color.

From this one can determine the maximum number of edges \( \text{wex}(n, K_m) \) in a graph of \( n \) vertices that has a \( K_m \)-WORM coloring.

**Theorem 6** The value \( \text{wex}(n, K_m) \) equals the maximum number of edges in a \( K_{(m-1)^2+1} \)-free graph.
Proof. Let $G$ be a graph on $n$ vertices with a $K_m$-WORM coloring. By Theorem 5, $G$ does not contain $K_{(m-1)^2+1}$ as a subgraph. Thus we $ex(n,K_m)$ is at most the Turán number $ex(n,K_{(m-1)^2+1})$. Further, the Turán graphs are complete $(m-1)^2$-partite graphs and thus, by the above discussion, have a $K_m$-WORM coloring. QED

Next we consider the complete bipartite equivalent of the above theorem. Trivially, the bipartite coloring of a bipartite graph is automatically an $F$-WORM coloring. So we focus on the maximum number of colors. In [8] we noted that: For $n \geq m \geq 2$, it holds that $W^+(K_{n,n},K_{1,m}) = 2m - 2$.

One can extend that as follows:

**Theorem 7** For $n \geq m \geq 2$, $W^+(K_{n,n},K_{m,m}) = n + m - 1$.

Proof. For the lower bound, color red $n - m + 2$ vertices in one of the partite set, and give every other vertex a distinct color. This uses $2n + 1 - (n - m + 2) = n + m - 1$ colors, and is a $K_{m,m}$-WORM coloring (since one partite set contains only $m - 1$ colors).

On the other hand, consider a $K_{m,m}$-WORM coloring of $K_{n,n}$ that uses at least $m + n$ colors. Then each partite set has at least $m$ colors that do not appear in the other partite set. This immediately gives a rainbow $K_{m,m}$. That is, $W^+(K_{n,n},K_{m,m}) \leq n + m - 1$. QED

5 Minimal Colorings

For graphs that do not have WORM colorings, one can ask how close to WORM can one get. We call a subgraph bad if it is rainbow or monochromatic, and define $B(G,F)$ as the minimum number of bad subgraphs isomorphic to $F$ in a coloring of $G$. In particular, $B(G,F) = 0$ means there is an $F$-WORM coloring of $G$.

Theorem 5 says that $B(K_n,K_m) \geq 1$ if $n \geq (m-1)^2 + 1$. One can ask what is the exact value of $B(K_n,K_m)$ in this case. We show that an optimal coloring uses $m - 1$ colors as equitably as possible.

**Theorem 8** Let $n = (m-1)k + j$ with $0 \leq j \leq m - 2$. Then $B(K_n,K_m) = j^{(k+1)} + (m - 1 - j)^{(k)}$.

Proof. Let an optimal coloring of $K_n$ be given. Assume that color $i$ is used $a_i$ times with $a_1 \leq a_2 \leq \ldots \leq a_r$. 

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Suppose that \( r \geq m \). Assume we recolor one vertex that has color 1 with color 2; let \( M \) denote the increase in the number of monochromatic \( K_m \)'s and \( R \) the decrease in the number of rainbow \( K_m \)'s. Then
\[
M = \binom{a_1 - 1}{m} - \binom{a_1}{m} + \binom{a_2 + 1}{m} - \binom{a_2}{m} \\
\leq \binom{a_2 + 1}{m} - \binom{a_2}{m} = \binom{a_2}{m-1} \leq a_2^{m-1},
\]
and
\[
R = a_2 \sum_{3 \leq i_1 \leq \ldots \leq i_{m-2} \leq r} a_{i_1} \ldots a_{i_{m-2}} \geq a_2 a_3 \ldots a_m \geq a_2^{m-1}.
\]
That is, \( M \leq R \). Hence the total number of bad \( K_m \)'s will not increase.

By repeating the argument, it follows that we may recolor all the vertices with color 1 and so reduce the total number of colors used. That is, we may assume that \( r \leq m - 1 \). In this case, rainbow \( K_m \)'s are impossible and thus we have only to minimize the number of monochromatic \( K_m \)'s. By convexity, optimality is achieved when \( m - 1 \) colors are used as equitably as possible. \qed

References


