Thoroughly Dispersed Colorings

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Abstract

We consider (not necessarily proper) colorings of the vertices of a graph where every color is thoroughly dispersed, that is, appears in every open neighborhood. Equivalently, every color is a total dominating set. We define \(td(G)\) as the maximum number of colors in such a coloring and \(FTD(G)\) as the fractional version thereof. In particular, we show that every claw-free graph with minimum degree at least 2 has \(FTD(G) \geq \frac{3}{2}\) and this is best possible. For planar graphs, we show that every triangular disc has \(FTD(G) \geq \frac{3}{2}\) and this is best possible, and that every planar graph has \(td(G) \leq 4\) and this is best possible, while we conjecture that every planar triangulation has \(td(G) \geq 2\). Further, although there are arbitrarily large examples of connected, cubic graphs with \(td(G) = 1\), we show that for a connected cubic graph \(FTD(G) \geq 2 - o(1)\). We also consider the related concepts in hypergraphs.

Keywords: Thoroughly dispersed coloring, total domination, coloring, transversal.

AMS subject classification: 05C69, 05C15

*Research supported in part by the South African National Research Foundation and the University of Johannesburg
1 Introduction

We consider (not necessarily proper) colorings of the vertices of a graph where every color is thoroughly dispersed, that is, appears in every open neighborhood. Chen et al. [7] called this the coupon coloring problem.

We define $td(G)$ as the maximum number of colors in such a coloring. Note that a color being thoroughly dispersed is equivalent to a color being a total dominating set (a set $S$ of vertices such that every vertex has a neighbor in $S$). Thus the parameter $td(G)$ is equivalent to the maximum number of disjoint total dominating sets, which Cockayne et al. [8] called the total domatic number. This parameter is now well studied. For example, Zelinka [29] showed that there are graphs with arbitrarily large minimum degree without two disjoint total dominating sets. Heggernes and Telle [14] showed that the decision problem to decide for a given graph $G$ if $td(G) \geq 2$ is NP-complete, even for bipartite graphs. In contrast, several researchers, such as Aram et al. [2], studied $td(G)$ for a $k$-regular graph $G$; in particular, Chen et al. [7] showed that such graphs have total domatic number at least $(1 - o(1))k/\ln k$. The related idea in hypergraphs, where every color appears in every edge, is called panchromatic coloring [20].

In this paper we consider the fractional analogue of the parameter $td(G)$. We define a thoroughly dispersed family $\mathcal{F}$ of a graph $G$ as a family of (not necessarily distinct) total dominating sets of $G$. We denote by $r_F$ the maximum number of times any vertex of $G$ appears in $\mathcal{F}$, and define the effective ratio of the family $\mathcal{F}$ as the ratio of the number of sets in $\mathcal{F}$ to $r_F$. The fractional total domatic number $FTD(G)$ is then defined as the supremum of the effective ratio taken over all thoroughly dispersed families. That is,

$$FTD(G) = \sup_{\mathcal{F}} \frac{|\mathcal{F}|}{r_F}.$$

Like other fractional parameters, one can show that the supremum can be achieved. For example, if we let $\mathcal{T}_G$ be the hypergraph with vertex set $V(G)$ and hyperedges all total dominating sets of $G$, then $FTD(G)$ is the fractional matching number of $\mathcal{T}_G$, and can be viewed as a linear program. (See Chapter 1 of [25] for further discussion.)

We remark that there have been a few papers on the ordinary domination equivalent. We let $dom(G)$ denote the domatic number of $G$, and so $dom(G)$ is the maximum number of disjoint dominating sets in $G$. We let $FD(G)$ denote the fractional domatic number, defined analogously as the fractional total domatic number; that is, the supremum of $|\mathcal{F}|/r_F$ taken
over all families $\mathcal{F}$ of dominating sets. The fractional domatic number seems to have been introduced by Suomela [26]. Recently, Abbas et al. [1] showed that: a $K_{1,6}$-free graph $G$ with minimum degree $\delta \geq 2$ has $FD(G) \geq 5/2$ except for some exceptions. See also [11].

We proceed as follows. In Section 2 we provide some preliminary results. In Section 3 we introduce the related concepts in hypergraphs. In Section 4 we consider claw-free graphs, and show that every claw-free graph with minimum degree at least 2 has fractional total domatic number at least $3/2$ and this is best possible. In Section 5 we consider planar graphs. Inter alia, we show that every triangular disc has fractional total domatic number at least $3/2$ and this is best possible; that a maximal outerplanar graph has two disjoint total dominating sets provided the order is not congruent to 2 modulo 4; and in general that a planar graph has total domatic number at most 4 and this is best possible. Finally in Section 6 we show that the fractional total domatic number of a connected cubic graph is at least $2 - o(1)$. Along the way we pose several open problems.

2 Preliminary Observations and General Properties

Recall that $\gamma_t(G)$ denotes the minimum size of a total dominating set. Since each set in a thoroughly dispersed family $\mathcal{F}$ of a graph $G$ has size at least $\gamma_t(G)$, by averaging we have that $r_F \geq |\mathcal{F}| \gamma_t(G)/n(G)$, implying that the effective ratio $|\mathcal{F}|/r_F$ is at most $n(G)/\gamma_t(G)$. If a thoroughly dispersed family $\mathcal{F}$ consists of a maximum number of disjoint total dominating sets of $G$, then $|\mathcal{F}| = td(G)$ and $r_F = 1$, implying that $FTD(G) \geq td(G)$. We state these observations formally as follows.

**Observation 1** If $G$ is an isolate-free graph of order $n$, then

$$td(G) \leq FTD(G) \leq \frac{n}{\gamma_t(G)}.$$ 

Equality occurs throughout the chain, for example, if $G = K_n$ and $n$ is even, or if $G = C_n$ and $n$ is a multiple of 4. More generally, equality occurs in the upper bound for all complete graphs and cycles. As a gentle introduction, we determine the fractional total domatic number of a cycle. Recall that $\gamma_t(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

**Observation 2** For $n \geq 3$, $FTD(C_n) = n/\gamma_t(C_n)$.
Proof. Let $G$ be the cycle $v_1v_2\ldots v_nv_1$. Let $S$ be an arbitrary minimum total dominating set of $G$, and so $|S| = \gamma_t(G)$. For $1 \leq i \leq n$, let $S_i = S + i = \cup_{j \in S} \{v_{i+j}\}$, where addition is taken modulo $n$. Each set $S_i$ is a minimum total dominating set of $G$. Let $\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$. Each vertex of $G$ appears in exactly $\gamma_t(G)$ of these sets, and therefore the effective total-ratio of the family $\mathcal{F}$ is $|\mathcal{F}|/r_\mathcal{F} = n/\gamma_t(G)$, implying that $FTD(G) \geq n/\gamma_t(G)$. The result now follows from the upper bound of Observation 1. \qed

For example, if we take $G$ as the 5-cycle, then $FTD(C_5) = 5/3$. However, $td(C_5) = 1$.

**Observation 3** If a graph $G$ has minimum degree $\delta \geq 1$, then $FTD(G) \leq \delta$.

Proof. Let $v$ be a vertex of minimum degree in $G$. Each set in a thoroughly dispersed family $\mathcal{F}$ of $G$ must contain a neighbor of the vertex $v$. So, by the Pigeonhole Principle, at least one neighbor of $v$ appears in at least $|\mathcal{F}|/\delta$ sets, and so $r_\mathcal{F} \geq |\mathcal{F}|/\delta$. This is true for every thoroughly dispersed family $\mathcal{F}$. \qed

On the other hand, we have the following:

**Observation 4** If a graph $G$ of order $n$ has minimum degree $\delta \geq 1$, then $FTD(G) \geq n/(n - \delta + 1)$.

Proof. Consider the collection $\mathcal{F}$ of all subsets $F$ of the vertex set of exactly $n - \delta + 1$ elements. Then every vertex has a neighbor in $F$; that is, $\mathcal{F}$ is a thoroughly dispersed family. Further, every vertex is in $\binom{n-1}{n-\delta}$ sets of $\mathcal{F}$. Thus, $\mathcal{F}$ has effective ratio $\left(\binom{n}{n-\delta+1}/\binom{n-1}{n-\delta}\right) = n/(n - \delta + 1)$. The result follows. \qed

Thus, for example, by Observations 3 and 4 we get the following observation.

**Corollary 5** (a) If a graph $G$ has minimum degree $\delta = 1$, then $FTD(G) = 1$.

(b) If a graph $G$ has minimum degree $\delta \geq 2$, then $FTD(G) > 1$.

We will show (Theorem 12) that there are graphs $G$ with arbitrarily large minimum degree with $FTD(G) < 1 + \epsilon$. Indeed, these are the graphs that Zelinka [29] provided as examples that have $td(G) = 1$ and arbitrarily large minimum degree.

We note that the union of two disjoint dominating sets is a total dominating set. Thus, it is immediate that $td(G) \geq \lfloor dom(G)/2 \rfloor$. But, in the case that the ordinary domatic number is odd, one can say slightly more:
**Theorem 6** If $G$ is an isolate-free graph, then $\text{FTD}(G) \geq \text{dom}(G)/2$.

**Proof.** If $D_1, \ldots, D_k$ are disjoint dominating sets, then $\mathcal{F} = \{ D_i \cup D_j : 1 \leq i < j \leq k \}$ is a thoroughly dispersed family. Every vertex appears in at most $k - 1$ sets; so $\text{FTD}(G) \geq \binom{k}{2}/(k - 1) = k/2$. \hfill \text{QED}

Equality occurs, for example, in complete graphs.

Note that $\text{FTD}$ is monotonic, in that it cannot decrease on the addition of edges.

We next show that the fractional total domatic number of the disjoint union behaves as expected.

**Theorem 7** If $G$ is the disjoint union of isolate-free graphs $G_1, G_2, \ldots, G_k$, then $\text{FTD}(G) = \min \{ \text{FTD}(G_1), \text{FTD}(G_2), \ldots, \text{FTD}(G_k) \}$.

**Proof.** It suffices to prove this for $k = 2$, as the full result follows by induction. We use the standard notation $[s] = \{1, 2, \ldots, s\}$.

For $\ell \in [2]$, let $\mathcal{F}_\ell = \{ T_{\ell,1}, \ldots, T_{\ell,k_\ell} \}$ be an optimal thoroughly dispersed family of $G_\ell$, and let $r_\ell = r_{\mathcal{F}_\ell}$. Then the collection $\mathcal{F} = \{ T_{1,i} \cup T_{2,j} : i \in [k_1], j \in [k_2] \}$ is a thoroughly dispersed family of $G$. If a vertex $v$ of $G_1$ appears in $r_v$ sets of $\mathcal{F}_1$, then $v$ appears in $r_v \cdot k_2$ sets of $\mathcal{F}$, and similarly with vertices of $G_2$. Thus, $r_\mathcal{F} = \max \{ r_1 k_2, r_2 k_1 \}$. So,

$$\text{FTD}(G) \geq \frac{k_1 k_2}{\max \{ r_1 k_2, r_2 k_1 \}} = \min \left\{ \frac{k_1}{r_1}, \frac{k_2}{r_2} \right\} = \min \{ \text{FTD}(G_1), \text{FTD}(G_2) \}.$$

Conversely, let $\mathcal{F}^* = \{ U_1, \ldots, U_k \}$ be an optimal thoroughly dispersed family of $G$. Then, for $\ell \in [2]$, the collection $\mathcal{F}_\ell^* = \{ U_i \cap V(G_\ell) : i \in [k] \}$ is a thoroughly dispersed family of $G_\ell$. Further, if each vertex appears at most $r$ times in $\mathcal{F}^*$, then $r_{\mathcal{F}_\ell^*} \leq r$. Thus

$$\text{FTD}(G_\ell) \geq \frac{k}{r} = \text{FTD}(G).$$

The two inequalities combined give the result. \hfill \text{QED}
3 Hypergraphs

We observed in the introduction that the fractional total domatic number of $G$ is also the fractional matching number of the hypergraph $T_G$. But hypergraphs also provide a more general setting for the parameter.

Recall that a subset $T$ of vertices in a hypergraph $H$ is a transversal (also called vertex cover or hitting set) if $T$ has a nonempty intersection with every edge of $H$. The transversal number $\tau(H)$ of $H$ is the minimum size of a transversal in $H$. See for example [4, 5, 6].

A hypergraph $H$ is 2-colorable if there is a 2-coloring of the vertices such that each hyperedge contains two vertices of distinct colors. In other words, there is no monochromatic hyperedge. So, the question of when a hypergraph has two disjoint transversals is the same as whether the hypergraph has a 2-coloring (also known as Property B), or whether a design has a blocking set. More generally, Kostochka and Woodall [20] defined a panchromatic $k$-coloring of a hypergraph as a coloring with $k$ colors such that every hyperedge contains each color. This is equivalent to a partition into $k$ disjoint transversals.

We denote by $\text{disj}_\tau(H)$ the disjoint transversal number of a hypergraph $H$, which is the maximum number of disjoint transversals in $H$.

Analogous to the fractional total domatic number, one can define the fractional disjoint transversal number. A transversal family $\mathcal{F}$ of a hypergraph $H$ is a family of transversals of $H$. Given a hypergraph $H$ and a transversal family $\mathcal{F}$, we define the effective transversal-ratio of the family $\mathcal{F}$ as the ratio of the number of sets in $\mathcal{F}$ over the maximum times $r_F$ any element appears in $\mathcal{F}$. The fractional disjoint transversal number $FDT(H)$ is the supremum of the effective transversal-ratio taken over all transversal families. That is,

$$FDT(H) = \sup_{\mathcal{F}} \frac{|\mathcal{F}|}{r_F}.$$

Analogous to Observation 1, we have the following bounds on the fractional disjoint transversal number.

**Observation 8** For every isolate-free hypergraph $H$ of order $n$,

$$\text{disj}_\tau(H) \leq FDT(H) \leq \frac{n}{\tau(H)}.$$

For example, if $H$ is the complete $k$-uniform hypergraph of order $n$, then $\tau(H) = n - k + 1$, and $FDT(H) = n/(n - k + 1)$, by considering the collection of all $n - k + 1$ element subsets as transversal family.
As another example, if we take $H$ to be the Fano plane and let $\mathcal{F}$ be the transversal family consisting of the seven edges of $H$, then $|\mathcal{F}| = 7$ and $r_\mathcal{F} = 3$, implying that $FDT(H) \geq 7/3$, with equality by the above observation. However, $\text{disj}_r(F_7) = 1$.

And similar to Observation 4 we have:

**Observation 9** If every edge of hypergraph $H$ of order $n$ has size at least $a$, then $FDT(H) \geq n/(n-a+1)$.

**Proof.** Consider the collection $\mathcal{F}$ of all subsets $F$ of the vertex set of exactly $n-a+1$ elements. Then every edge contains a vertex in $F$; that is, $\mathcal{F}$ is a transversal family. Further, every vertex is in $\binom{n-1}{n-a}$ sets of $\mathcal{F}$. Thus, $\mathcal{F}$ has effective ratio $\left(\binom{n}{n-a+1}/\binom{n-1}{n-a}\right) = n/(n-a+1)$. The result follows. \(\text{QED}\)

Analogous to Theorem 7, there is the following result on the fractional disjoint transversal number of the disjoint union of hypergraphs.

**Theorem 10** If $H$ is the disjoint union of isolate-free hypergraphs $H_1, H_2, \ldots, H_k$, then $FDT(H) = \min\{FDT(H_1), FDT(H_2), \ldots, FDT(H_k)\}$.

### 3.1 Connection

Associated with a graph $G$, one can define the open neighborhood hypergraph, abbreviated ONH, of $G$ as the hypergraph $\mathcal{O}N(G)$ whose vertex set is $V(G)$ and whose hyperedges are the open neighborhoods of vertices in $G$. Thus, if $H = \mathcal{O}N(G)$, then $V(H) = V(G)$ and $E(H) = \{N_G(x) : x \in V(G)\}$.

As Thomassé and Yeo [27] observed, a total dominating set in $G$ is a transversal in $\mathcal{O}N(G)$, and conversely. More generally, a thoroughly dispersed family of a graph $G$ is a transversal family of $\mathcal{O}N(G)$ and conversely. Thus, the fractional total domatic number of an isolate-free graph is precisely the fractional disjoint transversal number of its ONH.

**Observation 11** For every isolate-free graph $G$, $FTD(G) = FDT(\mathcal{O}N(G))$.

One can also consider the incidence graph $I(H)$ of a hypergraph $H$. (Recall that $I(H)$ is a bipartite graph with vertex set the vertices and edges of the hypergraph, where a vertex $v$
and edge $e$ are adjacent in $I(H)$ iff the vertex $v$ is contained in the edge $e$.) For example, the incidence graph of the complete $r$-uniform hypergraph $H$ on $n$ vertices with $n \geq 2r - 1$ is the standard example of a graph that does not have two disjoint total dominating sets but has arbitrarily large minimum degree, as shown by Zelinka [29].

In general, every connected bipartite graph has an ONH that consists of two components (see [16]). For example, consider the Heawood graph $G_{14}$. The ONH of the Heawood graph consists of two disjoint copies of the Fano plane, $F_7$. See Figure 1. Conversely, the Heawood graph is the incidence graph of the Fano plane. We note that $\gamma_t(G_{14}) = 6$ and $\tau(\mathcal{ON}(G_{14})) = 2\tau(F_7) = 6$. Further, by Observation 11 and Theorem 10, we have $\text{FTD}(G_{14}) = \text{FTD}(\mathcal{ON}(G_{14})) = \text{FTD}(F_7 \cup F_7) = \text{FTD}(F_7) = 7/3$. In contrast, the Heawood graph does not have two disjoint total dominating sets; that is, $\text{td}(G_{14}) = 1$.

![Figure 1. The Heawood graph and its open neighborhood hypergraph](image)

The subdivision graph of a graph $G$, denoted $S(G)$, is the graph obtained from $G$ by subdividing every edge of $G$ exactly once.

**Theorem 12** For $n \geq 3$, $\text{FTD}(S(K_n)) = n/(n - 1)$.

**Proof.** Let $G = S(K_n)$ with vertex set $V(K_n) \cup E(K_n)$. Then, the ONH of $G$ consists of two components. One, say $H_1$, has vertex set $V(K_n)$ and one, say $H_2$, has vertex set $E(K_n)$. The hypergraph $H_1$ is the graph $K_n$. We claim that $\text{FDT}(H_1) = n/(n - 1)$: the lower bound follows from Observation 9 while the upper bound follows from noting that $\tau(H_1) = n - 1$.

By Theorem 10 and Observation 11, it remains to show that $\text{FDT}(H_2) \geq n/(n - 1)$. For $n = 3$, the hypergraph $H_2$ is isomorphic to $H_1$. When $n$ is even, $\text{disj}_+(H_2) = n - 1$,
since the graph $K_n$ has a perfect 1-factorization in this case. It also follows that for $n$ odd, that $\text{disj}_\tau(H_2) \geq n - 2$. The result follows. QED

Note that $S(K_n)$ is also the incidence graph of $K_n$, if $K_n$ is thought of as a 2-uniform hypergraph. More generally, if we let $K^r_n$ denote the simple hypergraph of order $n$ with all $r$-element subsets as edges, we have the following result.

**Theorem 13** Let $k \geq 3$ be fixed and let $G_{k,n}$ be the incidence graph of the complete $k$-uniform hypergraph $K^r_n$ on $n$ vertices. Then, $\text{FTD}(G_{k,n}) = n/(n-k+1)$ for all $n \geq 2k$.

**Proof.** The ONH of $G_{n,k}$ consists of two components: the original hypergraph $K^r_n$ and its dual, say $H$. We claim that $\text{FDT}(K^r_n) = n/(n-k+1)$: the lower bound follows from Observation 9 while the upper bound follows from noting that $\tau(K^r_n) = n-k+1$. Thus, by Theorem 10 and Observation 11, it is sufficient to show that $\text{disj}_\tau(H) \geq 2$.

Indeed, $\text{disj}_\tau(H)$ is large. In particular, for $n$ a multiple of $k$, Baranyai’s theorem [3] says that $K^r_n$ has a 1-factorization and so $\text{disj}_\tau(H) = \binom{n-1}{k-1}$ in that case. For $n$ not a multiple of $k$, one can take the 1-factorization for $n'$, where $n'$ is the largest multiple of $k$ less than $n$, and then can find multiple disjoint hyperedges containing the remaining vertices. The result follows. QED

### 4 Claw-Free Graphs

We show next that a claw-free graph $G$ with minimum degree at least 2 has $\text{FTD}(G) \geq 3/2$.

We will need the following results.

Recall that if $G$ is a graph, $S$ a subset of vertices of $V(G)$, and $v$ a vertex of $G$, then an $S$-private neighbor of $v$ is a neighbor of $v$ that is not a neighbor of any other vertex of $S$. Cockayne, Dawes, and Hedetniemi [8] established the following property.

**Proposition 14** ([8]) If $S$ is a minimal total dominating set in graph $G$, then every vertex in $S$ has an $S$-private neighbor.

We will also need the result that every graph has two disjoint sets, one total dominating and one “half-total” dominating:
Theorem 15 ([15]) If $G$ is a graph with minimum degree at least 2, then, except for the 5-cycle, $G$ has disjoint dominating and total dominating sets.

We can now establish a lower bound on the fractional total domatic number of a claw-free graph with minimum degree at least 2.

Theorem 16 If $G$ is a claw-free graph with $\delta \geq 2$, then $\text{FTD}(G) \geq 3/2$.

Proof. As observed earlier, $\text{FTD}(C_5) = 5/3$. Hence, we may assume that $G$ is not a 5-cycle. By Theorem 15, the graph $G$ therefore has a partition $R$ and $B$ such that $R$ is a total dominating set and $B$ is a dominating set. By transferring vertices from $R$ to $B$ if necessary, we can assume $R$ is a minimal total dominating set.

We will construct three total dominating sets: one is $R$, and the other two $B_1$ and $B_2$ are supersets of $B$. We need to partition $R$ between $B_1$ and $B_2$. If this can be done, then the resulting thoroughly dispersed family $\mathcal{F}_G = \{R, B_1, B_2\}$ has the property that every vertex of $G$ appears in exactly two sets of $\mathcal{F}_G$, so that the effective ratio of $\mathcal{F}_G$ is $3/2$ and hence $\text{FTD}(G) \geq 3/2$.

Let $B'$ be the isolated vertices of $B$. The only vertices that do not have a neighbor in $B$ are those in $B'$. Now, for every vertex $x$ in $B'$ choose two neighbors $x_1$ and $x_2$ (necessarily in $R$) subject to the constraint that if possible one is an $R$-private neighbor of the other.

Now, construct an auxiliary graph $A$ with vertex set $R$ and two vertices in $A$ are adjacent if and only if they are the $\{x_1, x_2\}$ pair of some $x \in B'$. Observe that if $A$ is bipartite then we are done—that is, the 2-coloring is the desired partition of $R$—since every vertex in $B'$ is adjacent to a vertex in both partite sets.

So assume $A$ is not bipartite. Let $C: v_1v_2 \ldots v_kv_1$ be a shortest odd cycle in $A$. For $i \in [k]$, let $u_i$ be the vertex of $B'$ associated with the pair $\{v_i, v_{i+1}\}$ (where addition is taken modulo $n$). Thus, in $G$, $v_1u_1v_2u_2 \ldots v_ku_kv_1$ is a cycle. By claw-free-ness, every vertex in $R$ has at most two neighbors in $B'$. In particular, for each vertex $v_i$ of $C$, its neighbors in $B'$ are the two vertices $u_{i-1}$ and $u_i$ (where addition is taken modulo $n$). This implies, by claw-free-ness, that the $R$-private neighbors of a vertex $v \in V(C)$ all belong to $R$.

Because $C$ is odd, and $R$-private neighbors are unique or mutual, there must be a vertex $v$ in $C$ whose $R$-private neighbor $w$ is not on $C$. Renaming vertices, if necessary, we may assume that $v_1$ is such a vertex in $C$. We now consider the neighbors of $v_1$ in $B'$, namely $u_1$
and $u_k$. By the claw-free-ness of $G$, $u_1w$ or $u_kw$ is an edge of $G$. We may assume that $u_1w$ is an edge. Then by the choice of $x$’s neighbors, it must be that $v_2$ is another $R$-private neighbor for $v_1$ but it is on $C$, a contradiction. QED

We note that the lower bound of Theorem 16 is somewhat best possible: the graphs $K_3$ and $C_6$ have fractional total domatic number exactly $3/2$. Further, there are graphs with claws, such as the subdivision $S(K_4)$, that have smaller fractional total domatic number (see Theorem 12). Further, there are arbitrarily large connected $K_{1,4}$-free graphs with fractional total domatic number exactly $3/2$. For example, take an even number $k \geq 2$ of disjoint copies of the 6-cycle, let $(R, B)$ denote a bipartition of the resulting graph $kC_6$, and add a matching between the vertices of $R$ to make the graph connected. Since $C_6$ has three disjoint dominating sets, it follows from Theorem 6 that the resultant graph $G$ has $FTD(G) \geq 3/2$; on the other hand, every total dominating set of $G$ must contain at least two vertices of $R$ from each original 6-cycle in order to totally dominate the three vertices of $B$ that belong to that cycle, implying that $\gamma_t(G) \geq 2|R|/3$. Hence, if $\mathcal{F}$ is a thoroughly dispersed family of $G$, then, by averaging, we have that $r_x \geq |\mathcal{F}|/\gamma_t(G)/|R|$, implying that the effective ratio $|\mathcal{F}|/r_x$ is at most $|R|/\gamma_t(G) \leq 3/2$ and hence $FTD(G) \leq 3/2$. Consequently, $FTD(G) = 3/2$.

However, we believe that the lower bound of Theorem 16 should be improvable asymptotically. Perhaps it is true that if $G$ is a connected, claw-free graph with $\delta \geq 2$, then $FTD(G) \geq 2 - o(1)$ and/or there is a partition $(T_1, T_2)$ of the vertex set such that every vertex except possibly two has a neighbor in both $T_1$ and $T_2$. And maybe, if we require that $\delta(G) \geq 3$, then two disjoint total dominating sets are guaranteed in $G$.

5 Planar and Related Graphs

In this section, we consider the fractional total domatic number of planar graphs. Of course in general there are no lower bounds. So we focus on “dense” planar graphs.

5.1 Triangulated Discs

A triangulated disc is a (simple) planar graph all of whose faces are triangles, except possibly for the outer face. Matheson and Tarjan [22] showed that if $G$ is a triangulated disc, then $\text{dom}(G) \geq 3$. Hence, as an immediate consequence of Theorem 6 and the Matheson-Tarjan result, we have the following lower bound.
Theorem 17 If $G$ is a triangulated disc, then $\text{FTD}(G) \geq \frac{3}{2}$.

We remark that the lower bound of Theorem 17 is tight, in that there exist triangulated discs $G$ of arbitrarily large order satisfying $\text{FTD}(G) = \frac{3}{2}$. For example, consider the triangulated disc $G$ illustrated in Figure 2, where the shaded area consists of any maximal planar graph (or, equivalently, triangulation). Let $S$ be the set of three vertices on the outer face of $G$ that have degree at least 4. Since each set in a thoroughly dispersed family $\mathcal{F}$ contains at least two vertices of $S$, by averaging there is a vertex in $S$ that belongs to at least $2|\mathcal{F}|/3$ sets in $\mathcal{F}$, and so the effective ratio $|\mathcal{F}|/r_{\mathcal{F}}$ is at most $3/2$. Consequently, by Theorem 17, $\text{FTD}(G) = \frac{3}{2}$.

![Figure 2. A triangulated disc $G$ with $\text{FTD}(G) = 3/2$](image)

There are two extremal examples of triangulated discs: maximal planar graphs (where the outer face is a triangle) and maximal outerplanar graphs (where the outer face contains all vertices). We consider these next.

5.2 Maximal Outerplanar Graphs

The total domination number of maximal outerplanar graphs has been studied by several authors. In particular, Dorfling et al. [9] showed that, except for two exceptions, every maximal outerplanar graph with order $n \geq 5$ has total domination number at most $2n/5$. Since a maximal outerplanar graph has minimum degree 2, the most we can hope for is two disjoint total dominating sets (Observation 3). In this subsection we investigate when this occurs.

We will use the following construction. For a maximal outerplanar graph $G$ of order $k \geq 3$, define the graph $M(G)$ as follows. Start with $G$ and, for every edge $e = uv$ on
the outer boundary, add a new vertex $w_e$ with neighbors $u$ and $v$. Note that $M(G)$ has order $2k$ and is maximal outerplanar. For example, $M(K_3)$ is the Hajós graph or 3-sun. Another example of such a graph $M(G)$ is shown in Figure 3, where $G$ is the maximal outerplanar graph induced by the darkened vertices.

Figure 3. A graph $M(G)$

**Observation 18** If $G$ is a maximal outerplanar graph of odd order $k \geq 3$, then $M(G)$ does not have two disjoint total dominating sets.

**Proof.** The subhypergraph of $ON(M(G))$ induced by the open neighborhoods of all the $w_e$ is a 2-uniform hypergraph isomorphic to an $k$-cycle. Since $k$ is odd, such a cycle is not 2-colorable, which means that $ON(M(G))$ is not 2-colorable. That is, $M(G)$ does not have two disjoint total dominating sets. QED

For the proof of the next result we will need to recall a few concepts. The weak dual of a triangulated disc is the graph that has a vertex for every bounded face of the embedding, and an edge for every pair of adjacent bounded faces. It is known that if $G$ is a maximal outerplanar graph of order $n$, then the weak dual graph of $G$ is a tree of order $n - 2$ and maximum degree at most 3. Further, there is a canonical bijection between the edges of the weak dual and those edges of $G$ that are not on the outer cycle.

**Theorem 19** If $G$ is a maximal outerplanar graph of order $n \geq 4$ with $n$ not congruent to 2 modulo 4, then $G$ has two disjoint total dominating sets.

**Proof.** Consider an embedding of the maximal outerplanar graph $G$ in the plane. Let $C$ denote the outer hamiltonian cycle of $G$. The result is trivial if $n$ is a multiple of 4, since the cycle $C$ has the property.
If \( n \) is congruent to 1 modulo 4, then take the cycle \( C \) and consider a vertex \( v \) that has a neighbor \( w \) on \( C \) of degree 2 in \( G \). Give \( v \) and both its neighbors on \( C \) the same color; then alternate colors in pairs along the cycle \( C \). The only vertex whose neighbors on the cycle are the same color is \( v \); but \( v \) is adjacent to \( w \) and \( w \)'s other neighbor, which receive different colors. Thus, every vertex sees both colors, implying that \( G \) has two disjoint total dominating sets.

So assume \( n \) is congruent to 3 modulo 4, and so \( n \geq 7 \). Now, we claim that a maximal outerplanar graph has a chord \( e = uv \) such that the vertices \( u \) and \( v \) are distance 3 or 4 on the cycle \( C \). For, consider the weak dual \( D \) of \( G \). Recall, \( D \) is a tree of order at least 5 and maximum degree at most 3. Let \( t \) be a vertex of \( D \) that is not a leaf but has exactly one non-leaf neighbor \( t' \). Then the desired chord is the edge \( uv \) that is the dual of the edge \( tt' \), noting that if \( t \) has degree 2 in \( D \), then \( u \) and \( v \) are distance 3 on the cycle \( C \), while if \( t \) cannot be chosen to have degree 2, and therefore \( t \) has degree 3 in \( D \), then \( u \) and \( v \) are distance 4 on \( C \). For the coloring of \( G \), give \( u \) red, then the next two on the cycle blue, the next two red, and so on. Note that \( v \) gets color red, and so every vertex sees both colors. QED

The above result is best possible, in that for each \( n \) congruent to 2 modulo 4 there is a maximal outerplanar graph without two disjoint total dominating sets, namely the graphs \( M(G) \) defined above.

**Theorem 20** If \( G \) is a maximal outerplanar graph of order \( n \geq 6 \) congruent to 2 modulo 4, then \( FTD(G) \geq 2n/(n + 2) \), and this is best possible.

**Proof.** Since \( G \) is hamiltonian, \( FTD(G) \geq FTD(C_n) = 2n/(n + 2) \), by Observation 2.

To show that this bound is best possible, consider a maximal planar graph \( F \) of order \( k = n/2 \) and the graph \( M(F) \) defined above. Since \( n \geq 6 \) is congruent to 2 modulo 4, we note that \( k \geq 3 \) is odd. Consider a total dominating set \( T \) of \( M(F) \). Then, as above, the set \( T \) must contain one of every consecutive pair on the outer cycle of \( F \), so that \( T \) contains at least \((k + 1)/2 \) vertices of \( F \). Thus if \( \mathcal{F} \) is a thoroughly dispersed family of \( M(F) \), by averaging at least one vertex in \( V(F) \) appears in at least \( |\mathcal{F}|(k+1)/2k |V(F)| \) sets, and so \( r_{\mathcal{F}} \geq \frac{(k+1)}{2k} |\mathcal{F}| \). Therefore,

\[
\frac{|\mathcal{F}|}{r_{\mathcal{F}}} \leq \frac{2k}{k + 1} = \frac{2n}{n + 2}.
\]

As we have the matching lower bound, it follows that \( FTD(M(F)) = 2n/(n + 2) \). QED
We remark that the graphs $M(F)$ are not the only maximal outerplanar graphs $G$ that have $td(G) = 1$ and $FTD(G) = 2n/(n+2)$. Nevertheless, it can be shown that all examples share the feature of having an independent set of size $n/2$.

5.3 Upper Bounds for Triangulations

In this section we consider triangulations, where by triangulation we mean a simple graph embedded in some orientable surface such that every region is a triangle. We denote the average degree of a graph $G$ by $d_{av}(G)$.

Lemma 21 If $G$ is a triangulation of order at least 4, then $FTD(G) \leq d_{av}(G) - 1$.

Proof. Let $G$ be a triangulation with vertex set $V$ of order $n \geq 4$. Consider a thoroughly dispersed family $F = \{F_1, F_2, \ldots, F_k\}$ of $G$. For every $i \in [k]$ and every pair $u, v$ of vertices of $G$, define a weight function $g_i(u, v)$ as follows:

- $g_i(u, v) = 0$ if $u \not\in F_i$, or if $v$ is not a neighbor of $u$;
- $g_i(u, v) = 1/2$ if $u$ and $v$ lie in a triangle with some vertex $w$ where $u, w \in F_i$; and
- $g_i(u, v) = 1$ if $u \in F_i$, $v$ is a neighbor of $u$, but $u$ and $v$ have no common neighbor in $F_i$.

Consider any specific $i \in [k]$ and specific vertex $u \in F_i$. Since $F_i$ is a total dominating set of $G$, the vertex $u$ is guaranteed to have a neighbor $w$ in $F_i$. Thus, since $G$ has minimum degree at least 3, there are at least two choices for $v$ such that $g_i(u, v) = 1/2$. The remaining neighbors $v$ of $u$ all satisfy $g_i(u, v) \leq 1$. It follows that $\sum_{u \in V} g_i(u, v) \leq 1$. More generally, since $u$ appears in at most $r_F$ sets $F_i$,

$$\sum_{u \in V} \left( \sum_{i=1}^{k} \sum_{v \in V} g_i(u, v) \right) \leq \sum_{u \in V} r_F (\deg u - 1). \quad (1)$$

On the other hand, consider any specific vertex $v \in V$. For each $i \in [k]$, the vertex $v$ has a neighbor $u$ in $F_i$. If $u$ and $v$ are in a triangle with some other vertex $w$ in $F_i$, then $g_i(u, v) = g_i(w, v) = 1/2$; otherwise, $g_i(u, v) = 1$. Thus, $\sum_{u \in V} g_i(u, v) \geq 1$ for each $i \in [k]$, and so

$$\sum_{v \in V} \left( \sum_{i=1}^{k} \sum_{u \in V} g_i(u, v) \right) \geq \sum_{v \in V} \left( \sum_{i=1}^{k} 1 \right) = n \cdot k. \quad (2)$$

By Inequalities (1) and (2), it follows that

$$k \leq r_F \cdot \frac{1}{n} \sum_{u \in V} (\deg u - 1) = r_F (d_{av}(G) - 1).$$
Rearranged, this says that $|\mathcal{F}|/r_x = k/r_x \leq \overline{d}_{av}(G) - 1$, as required. \hspace{1cm} \text{QED}

If $G$ is a planar triangulation of order $n$, then $\overline{d}_{av}(G) = 6 - 12/n$. Thus, as an immediate consequence of Lemma 21, we have the following upper bounds.

**Theorem 22** A planar graph has total domatic number at most 4 and fractional total domatic number at most $5 - \frac{12}{n}$.

There do exist planar graphs $G$ with $td(G) = 4$. Computer search shows that the smallest such graph has order 16. For example, take the truncated tetrahedron and add a vertex inside each hexagonal face that is joined to all vertices on the boundary. Illustrated below (see Figure 4) is a spanning subgraph of this that still has four disjoint total dominating sets: the vertices labelled $i$ form a total dominating set for each $i \in [4]$.

![Figure 4. A planar graph $G$ with $td(G) = 4$](image)

For a general construction, one can take multiple copies of the above graph (or the graph shown in Figure 6) and connect up arbitrarily. (We remark that while this construction only produces graphs with restricted orders, it is possible to achieve intermediate orders.)

Note that Lemma 21 applies on all surfaces. For example, the average degree of a toroidal graph is at most 6. It follows that:

**Theorem 23** A toroidal graph has total domatic number at most 5.

We remark that there do exist toroidal graphs $G$ with $td(G) = 5$. Such an example is illustrated in Figure 5, where as usual the top and bottom dotted lines should be identified and similarly with the left and right dotted lines. Note that the vertices labelled $i$ form a total dominating set of $G$ for each $i \in [5]$. 

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By Theorem 22, a planar graph of order \( n \) has fractional total domatic number at most \( 5 - \frac{12}{n} \). The smallest planar graph \( G \) for which \( \text{FTD}(G) > 4 \) is illustrated in Figure 6.

![Figure 5. A toroidal graph \( G \) with \( td(G) = 5 \)](image)

![Figure 6. A planar graph \( G \) with \( \text{FTD}(G) = 21/5 \)](image)

### 5.4 A Construction

As examples for lower bounds, we will need the following constructions.

For a graph \( G \) 2-cell embedded on a surface (meaning every face is homeomorphic to a disc), the graph \( T(G) \) is obtained from the graph \( G \) by adding in each face of \( G \) a new vertex adjacent to every vertex on the face. In the case that \( G \) is planar, this is sometimes called the Kleetope of \( G \). We will call \( G \) the base graph.

In order to prove our next result, we will need the following structural lemma of triangulations.
Lemma 24 If $G$ is a triangulation of order at least 4 with set of vertices $V$ and set of faces $F$, then there is an injection from $V$ to $F$ such that every vertex is matched with a face on which it lies.

Proof. We use Hall’s theorem. Define for a set $S$ of vertices, the number $f(S)$ of faces incident with at least one vertex of $S$. It suffices to show that $f(S) \geq |S|$ for all $S \subseteq V$.

This follows by induction on the cardinality of $S$. Each vertex is incident with at least three faces, so $f(S) \geq 3$. Thus we may assume $|S| \geq 4$. There are two cases. If every face in $F$ is incident with $S$, then since a triangulation on $s$ vertices has at least $2s - 4$ faces, it follows that $f(S) \geq 2|S| - 4$. This is at least $|S|$ since $|S| \geq 4$. On the other hand, suppose there is a face in $F$ not containing a vertex of $S$. We now shade all faces in $F$ that contain vertices of $S$. There must be an edge $xy$ that separates a shaded and unshaded face. Let $v$ be the third vertex of the shaded face different from $x$ and $y$. We note that such a shaded face contains exactly one vertex of $S$, namely the vertex $v$. By the induction hypothesis, it holds that $f(S - \{v\}) \geq |S| - 1$, and $S$ is incident with at least one more face, so that $f(S) \geq |S|$, as required. \hfill \qedsymbol

Note that $T(G)$ has minimum degree 3, and thus $td(T(G)) \leq 3$.

Theorem 25 If $G$ is a triangulation of order at least 4, then $td(T(G)) = 3$ if and only if $G$ is 3-colorable.

Proof. Suppose that $td(T(G)) = 3$. Let $D_1, D_2, D_3$ be a partition of $V(T(G))$ into three vertex-disjoint dominating sets. Color the vertices of $G$ with $\{1, 2, 3\}$ by coloring each vertex $v$ with the index of the $D_i$ such that $v \in D_i$. Each vertex of $T(G)$ not in $G$ has degree exactly 3, and those three neighbors are in a different $D_i$. That is, every face of $G$ is properly colored by our coloring. Since every edge of $G$ is in a face, it follows that we have a proper 3-coloring. That is, $G$ is 3-colorable.

Conversely, suppose that $G$ has a proper 3-coloring. Then, each color class $C$ must appear in every face. Thus, every vertex of $T(G)$ not in $G$ has a neighbor in $C$, as does every vertex of $G$ that has a color different from $C$. It therefore suffices to show that for each vertex $v$ of $G$, we can find a new vertex to color with the same color as $v$. That is, we need a matching from the vertices of $G$ to the faces of $G$ that saturates the vertices of $G$. Such a matching exists by Lemma 24. \hfill \qedsymbol

We consider planar triangulations next.
**Theorem 26** If $G$ is a planar triangulation, then $td(T(G)) \geq 2$.

**Proof.** By the Four Color Theorem, or for example [24], one can 2-color the vertices of the base triangulation $G$ such that no face is monochromatic. Each color class $C$ in such a 2-coloring of $G$ totally dominates all vertices of $T(G)$ not in $G$, as well as all vertices of $G$ whose color is different from $C$. Further, by Lemma 24, we can match each vertex $v$ of $C$ with a new vertex $f_v$ of $T(G)$ not in $G$ and color $f_v$ with color $C$. In this way, every vertex of $C$ has a neighbor in $C$. Thus, both color classes form a total dominating set of $T(G)$.

QED

By Theorem 26, it follows that $FTD(T(G)) \geq 2$ for all planar triangulations $G$. We know of four examples of equality.

**Lemma 27** Let $G$ be one of the following four planar triangulations: $K_4$, the icosahedron, or one of the two graphs drawn in Figure 7. Then $FTD(T(G)) = 2$.

![Graphs whose Kleetopes have fractional total domatic number 2](image)

**Figure 7.** Graphs whose Kleetopes have fractional total domatic number 2

**Proof.** By Theorem 26, we have $td(T(G)) \geq 2$. So it suffices to show that $FTD(T(G)) \leq 2$.

Suppose firstly that $G = K_4$ and consider the Kleetope $T(G)$. In order to totally dominate the new vertices added in each face of the base graph $G$, all total dominating sets of $T(G)$ contain at least two vertices of $G$. Hence, if $\mathcal{F}$ is a thoroughly dispersed family of $T(G)$, then, by averaging over the four vertices in the base graph $G$, we have that $r_{\mathcal{F}} \geq 2|\mathcal{F}|/4$, implying that the effective ratio $|\mathcal{F}|/r_{\mathcal{F}}$ is at most 2 and hence $FTD(T(G)) \leq 2$. 19
Similarly, with a bit more effort (or a computer) it can be shown that for the graph of order 8 in Figure 7, and for the icosahedron, again every total dominating set of $T(G)$ contains at least half the vertices of the base graph $G$, so that it follows that every thoroughly dispersed family has effective ratio at most 2.

Such a property is not true for the graph of order 12 in Figure 7. Instead we used a computer to calculate the value of the parameter $FTD(T(G))$. QED

We briefly consider the case that $G$ is a quadrangulation.

**Lemma 28** If $G$ is a planar quadrangulation, then $td(T(G)) \leq 3$.

**Proof.** Suppose that $T(G)$ has four disjoint total dominating sets. As before, such a total domatic partition would, when restricted to $G$, be a proper 4-coloring of $G$. Each vertex of $G$ thus needs a neighbor of the same color, which must be one of the faces of $G$. But a facial vertex of $T(G)$ has four neighbors of different colors, so it can be the same color as only one neighbor. Thus we need a matching from $V(G)$ to $F(G)$ that saturates $V(G)$, but in a planar quadrangulation, $|V(G)| > |F(G)|$ and therefore there is no such matching. QED

Our second construction is as follows. For a triangulation $G$, the triangulation $U(G)$ is obtained from the graph $G$ by adding in each face $f$ of $G$ a new triangle $\{f_1, f_2, f_3\}$ each vertex of which has a different pair of neighbors on the boundary of the face. See Figure 8, where the white vertices are new. As before, we will call $G$ the base graph.

![Figure 8](https://example.com/figure8.png)

**Figure 8.** Construction of a triangulation $U(G)$

**Lemma 29** If $G$ is a planar triangulation, then $td(U(G)) = FTD(U(G)) = 3$. 20
Proof. By the result of Matheson and Tarjan [22], the base graph $G$ has three disjoint dominating sets, say $D_1$, $D_2$, and $D_3$. These can be extended into three disjoint total dominating sets of $U(G)$ as follows. For each face $f$ of $G$, add one vertex $f_i$ of $U(G)$ not in $G$ to each $D_i$ such that $f_i$ has a neighbor in $D_i$. Thus, $td(U(G)) \geq 3$.

On the other hand, consider a face $f$ of $G$. It is easy to verify that every total dominating set of $U(G)$ must contain at least two of the vertices from the 6-tuple consisting of the boundary of $f$ and the new triangle inside $f$. Thus, by averaging, the effective ratio of any thoroughly dispersed family of $U(G)$ is at most 3. QED

5.5 Lower Bounds for Triangulations

We now turn to lower bounds for triangulations. Unfortunately, we mostly have only open questions. Since every planar triangulation is a triangulated disc, Theorem 17 implies that every planar triangulation $G$ satisfies $FTD(G) \geq \frac{3}{2}$. We believe this lower bound can be improved significantly and pose the following conjecture.

Conjecture 30 If $G$ is a planar triangulation of order at least 4, then $td(G) \geq 2$.

We can establish the conjecture for a few cases.

Observation 31 If $G$ is a planar triangulation where every vertex has odd degree, then $td(G) \geq 2$.

Proof. This follows from the Four Color Theorem. If a vertex $v$ has odd degree in a triangulation, its neighborhood contains an odd cycle. Thus in a proper coloring, $v$ has neighbors of three different colors. Thus, the union of two color classes is a total dominating set. QED

Now, it is not true in general that if one has a proper 4-coloring of a planar triangulation, then one can just combine two color-classes to obtain a total dominating set. However, computer search suggests that:

Conjecture 32 Every planar triangulation with at least four vertices has a proper 4-coloring $(C_1, C_2, C_3, C_4)$ such that $C_1 \cup C_2$ and $C_3 \cup C_4$ are total dominating sets.
Equivalently, \( V(G) \) can be partitioned into two total dominating sets both of which induce a bipartite subgraph of \( G \).

Another case where Conjecture 30 (and Conjecture 32) holds is where the dual is hamiltonian.

**Observation 33** If \( G \) is a planar triangulation and the dual of \( G \) is hamiltonian, then \( \text{td}(G) \geq 2 \).

**Proof.** This uses a standard idea (as for example used by Tait). Consider a hamilton cycle of the dual of \( G \) as a closed curve \( C \) in the plane. Color vertices of \( G \) red if they are inside the curve \( C \), and blue if they are outside the curve. We claim that every vertex \( v \) has both a red and a blue neighbor. For, consider the cycle through \( N(v) \) as a closed curve \( D \). Then, the curve \( C \) must cut \( D \) (in at least two places). Say it cuts the edge \( xy \). Then \( x \) and \( y \) have different colors. QED

Now, the dual of a planar triangulation is a 3-connected cubic planar graph. The smallest such graphs that are not hamiltonian have 38 vertices. (See [19].) It can be checked that the duals of these nonhamiltonian graphs do have two disjoint total dominating sets. By the technique of Penaud [24], to prove Conjecture 30 it would be enough to show that: every 3-connected cubic planar graph has a 2-factor that does not include a facial cycle.

Now, there are planar triangulations that have total domatic number 2. For example, if we take \( T(G) \) for any planar triangulation with chromatic number 4. But there are also planar triangulations that have fractional total domatic number 2, though the four listed in Lemma 27 are the only we know of. This shows that Conjecture 30 if true is sharp.

We remark that the Kleetope construction shows that Conjecture 30 does not carry over to graphs embeddable on the torus or Klein bottle, as the Kleetope, \( T(K_5) \), of \( K_5 \) does not have two disjoint total dominating sets. That is:

**Observation 34** There exists toroidal triangulations \( G \) satisfying \( \text{td}(G) = 1 \).

If one imposes larger minimum degree, it appears even more can be said. We pose the following conjecture.

**Conjecture 35** If \( G \) is a planar triangulation with minimum degree at least 4, then \( \text{td}(G) \geq 3 \).
If true, this conjecture is sharp by the graphs $U(G)$ defined earlier. See Lemma 29.

We conclude this section with even more speculation. Perhaps it is true that every triangular disc with minimum degree at least 3 has two disjoint total dominating sets. (Computer search confirms this for small orders.) It is not true that every triangular disc with minimum degree at least 4 has three disjoint total dominating sets: the icosahedron minus a vertex is an example, and there is an example of order 10. But since computer search of small orders only finds these two examples, maybe there are only finitely many exceptions.

6 Cubic Graphs

In this section we consider the fractional total domatic number of cubic graphs. The question of which cubic graphs have two disjoint total dominating sets is well studied. For example, it is known that the Heawood graph is the smallest example without two disjoint total dominating sets. (For more information see for example McCuaig [23], or Gropp [13].) In contrast, Thomassen [28] showed that, for $r \geq 4$, every $r$-regular graph has two disjoint total dominating sets.

It is known [17] that a connected 3-regular 3-uniform hypergraph is almost 2-colorable, in that there is a 2-coloring that 2-colors all but one specified hyperedge. Similarly, a connected cubic graph has a 2-coloring such that each vertex, except possibly two, has both colors in its neighborhood. This enables us to provide a lower bound on the fractional parameters.

**Theorem 36** If $H$ is a connected 3-regular 3-uniform hypergraph on $n$ vertices, then $\text{FDT}(H) \geq 2n/(n+1)$.

**Proof.** Let $e$ be any hyperedge containing $v$ and let $H_v$ be the subhypergraph with $e$ removed from $H$. By the result of [17], the hypergraph $H_v$ has a 2-coloring, say $(A_v, B_v)$. If we add $v$ to whichever of $A_v$ and $B_v$ doesn’t already contain it, then both $A_v$ and $B_v$ are transversals of $H$.

Consider the collection $\mathcal{F} = \{A_v : v \in V(H)\} \cup \{B_v : v \in V(H)\}$. Each vertex $v$ is in at most $n+1$ of these sets. So $\mathcal{F}$ is a transversal family with effective transversal-ratio $2n/(n+1)$, as required. QED
By the connection with the ONH of $G$, it follows that a connected cubic graph $G$ of order $n$ has $FTD(G) \geq 2 - o(1)$. While this paper was under submission, the second author and Yeo [18] established what was conjectured in the original manuscript; namely

**Theorem 37** [18] If $G$ is a connected cubic graph, then $FTD(G) \geq 2$.

But the computer suggests that this might be slightly improvable; namely that maybe $FTD(G) > 2$ for all connected cubic graphs $G$ except those that have $\gamma_t(G) = n/2$ (which have $td(G) = FTD(G) = 2$). (See [16] for the description of such graphs.) It also remains open:

**Conjecture 38** If $G$ is a connected cubic graph, then $G$ has a thoroughly dispersed family of four sets such that every vertex is in at most two of these.

Or more generally, we ask:

**Question 39** Does every 3-uniform 3-regular hypergraph have four transversals with each vertex in at most two?

### 7 Open Questions

A lot remains to be ascertained. The two questions most frustrating to us are Conjecture 30, that every planar triangulation has two disjoint total dominating sets, and Conjecture 38, that every cubic graph has fractional total domatic number at least 2. We also wonder about the relationship between the total domatic number and its fractional counterpart. For example, how large can $FTD(G)$ be if $td(G) = 1$? The largest we know of is 7/3 from the Heawood graph.

**References**


