WORM Colorings of Graphs

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Rainbow Colorings

We color the vertices. A *rainbow* subgraph is one where every vertex has a different color.

Avoiding rainbow subgraphs is easy...
Monochromatic Colorings

A *monochromatic* subgraph is one where every vertex has the same color.

Avoiding monochromatic subgraphs is easy...
Every coloring of $K_5$ must have either rainbow or monochromatic $K_3$. (Pigeonhole. . . )

But we can color $K_4$ without. (Two colors each used twice.)
Fix graph $F$. An $F$-WORM coloring of graph $G$ is a coloring of the vertices such that $G$ is Without a Rainbow or Monochromatic subgraph isomorphic to $F$. 
If graph $G$ has $F$-WORM coloring, we define:

- $W^+(G, F)$ is maximum number of colors in $F$-WORM coloring of $G$
- $W^-(G, F)$ is minimum number of colors in $F$-WORM coloring of $G$. 
Warm-Ups

- We assume $F$ has at least three vertices. ($K_2$ and $2K_1$ always rainbow/monochromatic.)
- $W^+(K_4, K_3) = W^-(K_4, K_3) = 2$.
- If $G$ is bipartite, a proper 2-coloring is $F$-WORM. (Can generalize...)
Bujtás et al.: 3-Consecutive $C$-colorings

Axenovich et al.: Mainly questions for edge colorings.
Sample result.

**Theorem.** $W^{-}(P_n, P_3) = 2$

**Theorem.** $W^{+}(P_n, P_3) = \lceil (n + 1)/2 \rceil$
**Theorem.** A graph has a $P_3$-WORM coloring if and only if it has a $P_3$-WORM coloring using only 2 colors.

Multiple proofs. . .
Proof Part 1

The monochromatic edges form a matching:

\[\text{vegasWORM: 11}\]
Proof Part 2

The rainbow edges form a bipartite graph:
Proof Part 3

Bipartite coloring plus matching is $P_3$-WORM.
By above, $P_3$-WORM colorings exists if and only if one with 2 colors.

Such a coloring is a 2-coloring such that each vertex has at most one neighbor of the same color. That is, *a coloring with defect 1*. (Known NP-hard: R. Cowen.)
Failed Conjecture

Not guaranteed a $P_3$-WORM coloring using $j$ colors for $2 < j < W^+(G, P_3)$. This graph has $P_3$-WORM colorings with exactly 2 or 4 colors:
For cubic, $P_3$-WORM coloring exists by Lovász. Conjecture that $W^-(G, P_3) \leq n/4 + 1$ for all cubic $G$. Here is extremal.
For Maximal Outerplanar Graphs

Can show: If exists, can only use 2 colors.

Can show: Characterization.
Theorem. If graphs $G$ and $H$ have a $P_3$-WORM coloring, then $W^+(G \square H, P_3) = 2$.

Proof idea: start at any vertex and grow coloring around it.
And Now For Something Completely Non-Different

Some results generalize to other stars, paths, trees etc.

But the fundamental: if exists then with 2 colors, does not.
Sample results:

- For outerplanar: exists (arboricity 2); formula for $W^+$
- For cubic graphs: exists (by Lovász again); bounds for $W^+$
Choose set of graphs $\mathcal{F}$ and define a $\mathcal{F}$-WORM coloring.

Define $B(G, F)$ as minimum number of monochromatic or rainbow copies of $F$; measures “how far away” $G$ is from having a WORM coloring.
Your Moment of Oz

Is the yellow-brick road “monochromatic” or “over the rainbow”?