Distance and Connectivity Measures in Permutation Graphs

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Abstract

A permutation graph G^π of a graph G (or generalized prism) is obtained by taking two disjoint copies of G and adding an arbitrary matching between the copies. For the parameters diameter, radius, average distance, connectivity and edge-connectivity, we compare the values of the parameter for G^π and G. In particular we show that if G has no isolates and is not $2K_k$ for k odd, then there exists a permutation graph of G with edge-connectivity equal to its minimum degree.

Key words: permutation graph, generalized prism, augmenting, connectivity

1 Introduction

In [2] Chartrand and Harary introduced permutation graphs. For a graph G and a permutation π of $V(G)$, the ***permutation graph*** G^π is defined by taking two disjoint copies of G and adding a matching joining each vertex v in the first copy to $\pi(v)$ in the second copy. Some have called these ***generalized***

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prisms. Examples of these graphs include hypercubes, prisms, cycle permutation graphs and some generalized Petersen graphs. The term permutation graph has also a meaning in the world of perfect graphs and their kin, but the two kinds are unrelated.

Many authors have considered properties of permutation graphs or of particular classes of permutation graphs. Properties which have been examined include planarity and its variants, chromatic number and its variants, hamiltonicity and its variants and vulnerability measures such as toughness, integrity and binding number, as well as questions about isomorphism. In particular, connectivity and edge-connectivity were studied in [7–11] while diameter and distance were studied in [4–6].

One phenomenon that has been observed is that the identity permutation is extremal with respect to some of these parameters. It is the purpose of this paper to survey how various parameters compare for G and G^π. We provide bounds for the parameters radius, diameter, total distance, connectivity and edge connectivity, especially the minimum and maximum value for the permutation graph relative to the original graph. Along the way we will see more parameters for which the identity permutation gives the maximum or minimum value of the parameter.

2 Notation

The order, number of edges and minimum degree of a graph G are denoted by $p(G)$, $q(G)$ and $\delta(G)$ respectively. The diameter, radius, connectivity and edge-connectivity are denoted by $\mathrm{diam}(G)$, $\mathrm{rad}(G)$, $\kappa(G)$ and $\lambda(G)$ respectively. The neighbourhood of a vertex v is denoted $N(v)$ and is the set of vertices adjacent to v.

We will denote the two copies of G in the permutation graph G^π by G_1 and G_2, and sometimes call them the layers. The edges between G_1 and G_2 we call ***cross edges***. For a vertex v in G_1 we will use the notation v' to denote its neighbour (specified by $\pi(v)$) in G_2.

The identity permutation is denoted by id, and the permutation graph G^{id} is the cartesian product $G \times K_2$. For any permutation graph G^π, we define a ***single twist*** of it as the permutation graph resulting from changing π by a single transposition.

The following construction provides some extremal permutation graphs. If G is a graph with m^2 vertices and $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ is a partition of the vertex set into m sets of cardinality m, then a ***spread permutation*** is any permutation π such that for every i and j there is a vertex $v \in A_i$ such that $\pi(v) \in A_j$. We will use two specific cases in particular. First if $G = mK_m$ then the partition $\mathcal{A}$ will consist of the cliques, and if G is the complete m-partite graph with m vertices in each partite set, then $\mathcal{A}$ will consist of the partite sets. For these specific G we will simply speak of the spread permutation graph of G.

3 Diameter

Here we consider how the diameters of G and G^π can compare. Gu observed the following:

Observation 1 *[4] For all graphs G and permutations π,* $\operatorname{diam}(G^\pi) \leq \operatorname{diam}(G^{id}) = \operatorname{diam}(G) + 1$.

Gu [4] showed that if G has at least 3 vertices, then there is at least one other π for which $\operatorname{diam}(G^\pi) = \operatorname{diam}(G) + 1$.

A sufficient condition for $\operatorname{diam}(G^\pi) < \operatorname{diam}(G) + 1$ is that at most half the vertices of G have eccentricity equal to $\operatorname{diam}(G)$. For some graphs, the permutation graph for G can have much lower diameter than G. There are a few limits on this though.

Jia [6] constructed graphs with large diameter whose permutation graphs have diameter 5. Independently, Gu [5] constructed graphs with large diameter whose permutation graphs have diameter 4. In fact, Gu constructed for some integers (l, d) a graph with diameter l which has a prism with diameter d. We show that the list is exhaustive:

Theorem 2 *There exists a graph G and permutation π such that* $\operatorname{diam}(G) = l$ *and* $\operatorname{diam}(G^\pi) = d$ *iff $d = 2$ and $l \in \{1, 2\}$; $d = 3$ and $l \in \{2, 3, 4, 5\}$; or $d \geq 4$ and $l \geq d - 1$.*

PROOF. The existence is from [5]. Gu [4] also observed that if $\operatorname{diam}(G^\pi) = 2$, then $\operatorname{diam}(G) \leq 2$. It remains only to show that if $\operatorname{diam}(G) \geq 6$ then $\operatorname{diam}(G^\pi) \geq 4$.

Suppose that the graph G has diameter at least 6 but G^π has diameter 3. Consider a vertex v of G of maximum eccentricity. Let W be the set of vertices of G at distance at least 4 from v, and let w be a vertex of G farthest from v. By slight abuse of notation, we may think of v, W and w as being in G_1.

A shortest path from a vertex $u \in W$ to v must use a cross edge. In fact it must use both the cross edges uu' and vv', and so $u' \in N(v')$.

Now consider a vertex x' in G_2 such that its distance from both v' and w' is at least 3. (It exists, since otherwise every vertex is within distance 2 of one of $\{v', w'\}$, which are adjacent, and so $\mathrm{diam}(G) \leq 5$.)

Consider the shortest path from x' to w in G^π. A length-3 path P can use only one cross edge, say aa'. But since the vertex a is within distance 2 of w as measured in G_1, a is in W and so a' is a neighbour of v'. That means a' is at least distance 2 from x' as measured in G_2, and thus aa' is the last edge of P; i.e., $a = w$. But that contradicts the choice of x. □

For specific graphs, we will consider first the complete multipartite graph. The above observation shows that any resultant permutation graph has diameter at most 3.

Theorem 3 *There is a permutation π such that* $\mathrm{diam}(K_{m_1,m_2,\ldots,m_t}) = 2$ *iff there exists a simple bipartite graph on $2t$ vertices where each partite set has degree sequence $m_1, m_2, \ldots, m_t$.*

PROOF. Assume G has partite sets $A_1, \ldots, A_t$. Let B_{ij} denote the set of vertices of G_1 in A_i whose neighbours in G_2 are in A_j. Say $|B_{ij}| = b_{ij}$.

Consider a vertex v in B_{ij}. Then v reaches all vertices in G_1 in at most two steps. Further, via v', the vertex v reaches all vertices in G_2 in at most two steps, except possibly for the vertices of $A_j - \{v'\}$. In fact, the vertices at distance 3 from v are precisely those vertices of $A_j - \{v'\}$ in G_2 whose neighbours in G_1 are in A_i. It follows that G^π has diameter 2 iff all $b_{ij} \leq 1$.

In particular, if we contract each partite set of G^π to a single vertex, then the result is simple iff G^π has diameter 2. So a necessary and sufficient condition for π to exist with diameter 2 is that there is a bipartite graph with t vertices in each partite set where each partite set has degree sequence $m_1, m_2, \ldots, m_t$. Bigraphic/bigraphical sequences are discussed in exercises in [3,12]. □

For example, the complete bipartite graph $K_{a,b}$ has a permutation graph with diameter 2 iff both a and b are at most 2. (To get diameter 2 for C_4, one needs to twist the hypercube.)

Another special case is the path on n vertices. We use a version of the butterfly construction.

Theorem 4 *For integer n, the minimum diameter of P_n^π over all permutations π is $\Theta(\log n)$.*

PROOF. For $k \geq 2$, we define the permutation graph F_k recursively. Start with two paths with $2^k - 2$ vertices each, each numbered from 1 up to $2^k - 2$. Join each vertex 1 to the vertex 2^{k-1} in the other path. Then recursively construct F_{k-1} on the two pieces of length $2^{k-1} - 2$ remaining (vertices 2 up to $2^{k-1} - 1$ and vertices $2^{k-1} + 1$ up to $2^k - 2$). If $k = 2$ then $F_2 \cong C_4$.

The graph F_k has the property that every vertex is distance at most $2k - 3$ from one of the leftmost endpoints (numbered 1). For, each left endpoint of the F_{k-1} subgraphs can reach a left endpoint of F_k in at most 2 steps, and the base case of the induction is $k = 2$ which is the 4-cycle.

Now, define the permutation graph F'_k by taking F_k and extending each path to the left by one vertex and making the two new vertices adjacent. The graph F'_k has diameter at most $2(2k - 3) + 3 = 4k - 3$. That is, F'_k is a permutation graph of the path on $n = 2^k - 1$ vertices, and has diameter about $4 \log_2 n$. □

By a similar construction it holds that the minimum diameter of the permutation graph of the cycle on n vertices is $\Theta(\log n)$.

Jia [6] proved that for any fixed r it holds that for any r-regular graph G_r there exists a permutation such that

$$\mathrm{diam}(G_r^\pi) \leq O\left(\frac{(\log n)^3}{(\log \log n)^2}\right)$$

Jia pointed out that $\mathrm{diam}(G_r^\pi) \geq c_r \log n$ since it has fixed degree.

A natural question is about interpolation. Suppose that the minimum diameter of a permutation graph for G is f. Is there for all integers s between f and $\mathrm{diam}(G) + 1$ a permutation π such that $\mathrm{diam}(G^\pi) = s$?

The idea of changing π slowly is the obvious attempt. But note that a single twist of G^π can greatly affect its diameter. For example, let n be a multiple of 4 and let G be the graph formed from the cycle on n vertices by adding one chord joining diametrically opposite vertices. For the identity permutation $\mathrm{diam}(G^{id}) = n/2 + 1$. But the permutation graph where the neighbours of antipodal vertices farthest from the chord are swopped has diameter $n/4 + O(1)$.

4 Radius

The results for radius are similar to those for diameter.

Theorem 5 *For all graphs G and permutations π it holds that*

$$\mathrm{rad}(G^\pi) \le \mathrm{rad}(G^{id}) = \mathrm{rad}(G) + 1$$

PROOF. The upper bound follows by noticing that a vertex in G can have its eccentricity increase by at most 1 in G^π. $\square$

It is immediate that if $\mathrm{rad}(G^\pi) < \mathrm{rad}(G)$ then $\mathrm{rad}(G) \ge 3$.

Theorem 6 *There are graphs G with arbitrarily large radius but* $\mathrm{rad}(G^\pi) = 3$.

PROOF. Fix a positive integer r. Then construct a graph G as follows. Start with the path on $2r+1$ vertices. Then expand the central vertex to be a clique on $2r+1$ vertices. The result has radius r.

Then choose π such that every vertex in G_1 not in the central clique has a neighbour in the central clique of G_2 and every vertex in G_2 not in the central clique has a neighbour in the central clique of G_1. Then there is a vertex v in the central clique of G_1 whose neighbour v' in G_2 is in the central clique of G_2. We claim that v has eccentricity at most 3: each vertex of G_2 is either adjacent to v' or adjacent to the central clique of G_1 and so is distance at most 2 from v. $\square$

It is easy to see that we can replace 3 in the above theorem by anything larger. So in summary:

Theorem 7 *There exists a graph G and permutation π such that* $\mathrm{rad}(G) = l$ *and* $\mathrm{rad}(G^\pi) = r$ *iff $r = 2$ and $l \in \{1, 2\}$; or $r \ge 3$ and $l \ge r - 1$.*

We turn next to special cases. The path and cycle are similar to before.

Theorem 8 *If G is a complete multipartite graph, then there exists a permutation π such that* $\mathrm{rad}(G^\pi) = 2$.

PROOF. Consider the smallest partite set A_1. Choose a permutation such that there is one edge vv' beween the copies of A_1 in the two layers. This is possible since $|A_1| \leq p(G)/2$. In the terminology of the proof of Theorem 3, $b_{11} = 1$ and v has eccentricity 2. □

5 Distance

The ***total distance*** (also known as the distance or transmission) of a graph G is the sum over all pairs of vertices u, v the distance $d(u, v)$ between u and v. It is denoted by $d(G)$.

Theorem 9 *For all graphs G and permutations π,*

$$d(G^\pi) \leq d(G^{id}) = 4d(G) + p(G)^2.$$

PROOF. The sum of the distances between vertices inside G_1 is at most $d(G)$ and similarly for G_2. For the distances between vertices x and y' in different layers, consider a path whose first edge is a cross edge and uses no other cross edge. It follows that

$$\sum_{x,y'} d(x, y') \leq \sum_{x,y'} 1 + d(x', y') = p^2 + 2d(G),$$

which completes the proof. □

Earlier, Jia [6] observed that for nontrivial graphs G the average distance obeys:

$$ave(G^\pi) \leq ave(G) + 1/2.$$

In fact, the inequality is strict (as can be deduced, for example, from above).

As a special case, consider the complete multipartite graphs G. Recall that any permutation graph for G has diameter at most 3. Also, any permutation graph for G has the same number of edges. Therefore, minimising the total distance $d(G^\pi)$ is equivalent to minimising the number of pairs of vertices at distance 3. In fact, the number of pairs at distance 1 is $2q(G) + p(G)$, and in the notation of the proof of Theorem 3, the number of pairs at distance 3 is $\sum_{i,j} b_{ij}(b_{ij} - 1)$, and a quick calculation shows that

$$d(G^\pi) = 4p(G)^2 - 4p(G) - 2q(G) + \sum_{i,j} b_{ij}^2.$$

So we have an optimization problem:

minimise $\sum_{i,j} b_{ij}^2$
s.t. b_{ij} nonnegative integers
for all i $\sum_j b_{ij} = k_i$
for all j $\sum_i b_{ij} = k_j$

In the special case where each of the t partite sets has size kt for some k, then the optimum has $b_{ij} = k$ for each i, j with the edges leaving one partite set evenly spread amongst the partite sets of the other layer. But in general the optimisation does not admit a clean formula.

For a path or cycle on n vertices we saw earlier that the diameter of the permutation graph can be $O(\log n)$, and so the total distance of the permutation graph is at most $O(n^2 \log n)$.

6 Connectivity

Connectivity and edge-connectivity were studied by Piazza et al. [8–11] and Lai [7]. We consider here the question of the minimum and maximum value of connectivity.

In [10] a lower bound for the connectivity of G^π is established in terms of the connectivity of G and a parameter called $U(G)$. Later, Lai [7] observed that in fact $U(G) = \delta(G) + 1$. Thus one may summarise the situation as follows:

Theorem 10 *[7] For all graphs G and permutations π,* $\min(2\kappa(G), \delta(G) + 1) = \kappa(G^{id}) \leq \kappa(G^\pi) \leq \delta(G) + 1$.

An interesting question is to characterise those graphs G which have a permutation π such that $\kappa(G^\pi) = \delta(G) + 1$. If G is sufficiently connected relative to its minimum degree, then this is guaranteed for all permutations, but even when G has low connectivity it seems likely that such a permutation π exists.

For the special case of $\delta(G) = 0$, the following result (which is true for all G) provides the answer.

Observation 11 *There exists a π such that G^π is connected iff G has at most $(p(G) + 1)/2$ components.*

PROOF. Suppose G has k components and p vertices. If we take G^π and contract each component in each copy of G to a single vertex, the result has $2k$ vertices and p edges. To be connected it is necessary that $p \geq 2k - 1$.

We prove by induction that if $k \leq (p+1)/2$ then there exists a permutation graph that is connected. If $k < (p+1)/2$, then remove any non-cut-vertex and induct. If $k = (p+1)/2$, then there is an isolated vertex x and a non-cut-vertex y. Induct on $G - \{x, y\}$ and add cross edges joining each copy of x to a copy of y. □

For $\delta \in \{1, 2\}$ we show there is a permutation graph with maximum connectivity.

Theorem 12 *Suppose $\delta(G) = 1$. Then there exists a permutation π such that $\kappa(G^\pi) = 2$.*

PROOF. It suffices to prove this for a graph G that is minimal with respect to $\delta(G) = 1$. So G is the disjoint union of stars. Call a vertex large if it has degree 2 or more. The proof is by induction on the number of vertices. If there is no large vertex, then G^π can easily be made hamiltonian. So we may assume there is at least one large vertex.

Let u be a neighbour of a large vertex and let $G' = G - \{u\}$. Then, by the inductive hypothesis, G' has a permutation graph G'^π which is 2-connected. To this graph add back the vertices corresponding to u in each layer. Then join uu'. The result is a permutation graph of G, and it can be checked that it is 2-connected. □

We will need the following observation which is easily proven.

Observation 13 *Let G be a 3-connected graph. If a, b, c, d are distinct vertices, then the result of adding vertices x and y with neighbourhoods $N(x) = \{a, b, y\}$ and $N(y) = \{c, d, x\}$, and deleting edges ab and/or cd if they exist, is 3-connected.*

Theorem 14 *Suppose $\delta(G) = 2$. Then there exists a permutation π such that $\kappa(G^\pi) = 3$ unless $G = 2K_3$.*

PROOF. It suffices to prove this for a graph G that is minimal with respect to $\delta(G) = 2$. So in G every edge is incident with a vertex of degree 2.

Call a vertex ***large*** if it has degree 3 or more. The proof is by induction on the order. The base case is handled by Case 3.

CASE 1: *There are two adjacent vertices x and y of degree 2 which are not in a triangle.* Say the other neighbour of y is z. Then form G' by deleting

y and adding the edge xz. If $G' \ncong 2K_3$, then by the inductive hypothesis, there is a 3-connected permutation graph for G'. Into this reintroduce the two copies of y by subdividing the two copies of the edge xz, and add the edge yy'. The result is a permutation graph for G, and by the above observation it is 3-connected. If $G' \cong 2K_3$, then G is the disjoint union $C_4 \cup C_3$ and the existence of a suitable permutation can be checked.

Case 2: *There is a degree-2 vertex v with two large neighbours.* Then form $G' = G - v$. If $G' \ncong 2K_3$, then by the inductive hypothesis, there is a 3-connected permutation graph for G'. Then reintroduce the two copies of v and add the edge vv'. By the above observation, the result is a 3-connected permutation graph for G. If $G' \cong 2K_3$, then there is only one possibility for G (by minimality), and it is easily checked that there is a suitable permutation graph which is 3-connected.

Case 3: *Otherwise.* So each component of G has at most one large vertex, and every cycle is a triangle. Thus the edge-set of G can be partitioned into triangles. If there are at most two triangles, then G is either K_3 or the bow-tie $K_1 + 2K_2$, and the latter is easily verified. So we may assume there are at least three triangles.

Label the triangles $T_1, \ldots, T_m$. For triangle T_i, let u_i and v_i be small vertices, and let x_i be the remaining vertex. Then construct G^π as follows. Start with the two copies of G. Let $X = \bigcup_i \{x_i, x'_i\}$ and $Y = \bigcup_i \{u_i, v_i, u'_i, v'_i\}$. Add cross edges within X such that each x_i is adjacent to x'_i (note that if $x_i = x_j$ then $x'_i = x'_j$). Then add $2m$ cross edges so that Y induces a cycle C. Specifically, if m is odd, then add $u_i v'_{i-1}$ and $v_i u'_{i+1}$ for $1 \le i \le m$ (with arithmetic modulo m). If m is even, then the cross edges are as in the case m odd, except that $u_1 u'_{m-1}$ and $v_{m-2} v'_m$ replace $u_1 v'_m$ and $v_{m-2} u'_{m-1}$, and $u_{m-2} u'_1$ and $v_m v'_{m-3}$ replace $u_{m-2} v'_{m-3}$ and $v_m u'_1$.

To see that the result is 3-connected, note that every vertex is either on C or adjacent to C. So if the removal of two vertices disconnect G^π, it must disconnect C. But then all of X remains, and C was chosen so that for most i the pairs $u_i v_i$ and $u'_i v'_i$ are almost diametrically opposite on C, so that X holds the two pieces of C together. □

It is an open question whether there are similar results for higher minimum degree.

If we turn to interpolation, we note that a single twist of G^π can affect the connectivity by at most 2. The graph $G = 2K_n$, for n even, shows that there is no better interpolation result, as $\kappa(G^\pi)$ is always even.

7 Edge-Connectivity

For edge-connectivity Lai proved similar bounds:

Theorem 15 *[7] For all graphs G and permutations π,* $\min(2\lambda(G), \delta(G)+1) = \lambda(G^{id}) \leq \lambda(G^{\pi}) \leq \delta(G)+1$.

We turn to the question of which graphs G have a permutation π such that $\lambda(G^{\pi}) = \delta(G)+1$. The special case of $\delta(G) = 0$ is covered by Observation 11. The next theorem shows that for $\delta(G) \geq 1$ a permutation graph with the desired edge-connectivity almost always exists.

Theorem 16 *Let G be a graph without isolates. Then there exists a permutation π such that $\lambda(G^{\pi}) = \delta(G)+1$ unless $G = 2K_k$ for some odd k.*

For $G = 2K_k$ it is easily checked that $\lambda(G^{\pi}) \leq k-1$ always. The existence proof uses the deep results of Bang-Jensen, Gabow, Jordán and Szigeti [1] about edge-connectivity augmentation with partition constraints.

We will need a few concepts from that paper. Fix an integer $k \geq 2$. For a multigraph G and a set A of vertices, the ***edge-boundary*** $d_G(A)$ is the number of edges with one end in A and one end outside A. We say that A is ***k-expansive in*** G if the edge-boundary $d_G(S)$ of every proper subset S of vertices of A is at least k. So a graph G is k-edge-connected iff $V(G)$ is k-expansive.

Consider a multigraph H^s with vertex set $V \cup \{s\}$ where s is a special vertex and V is k-expansive. If sr and st are edges, then a ***splitting off*** of the pair means replacing the pair by a new edge rt. Such a pair is ***k-admissible*** if after the splitting the set V is still k-expansive. Suppose further that there is a fixed partition $\mathcal{P}$ of V. An admissible pair sr and st is ***k-allowed*** if r and t are in different sets of $\mathcal{P}$.

Bang-Jensen et al. characterised when every edge incident with s can be split off using k-allowed pairs (which we call a ***complete k-allowed splitting***). For a set X we denote by $d(s, X)$ the number of edges joining s to X.

Theorem 17 ([1], Theorem 3.6) *Let $k \geq 2$ and H^s be as above with V k-expansive in H^s and suppose s has even degree $d(s)$. There is a complete k-allowed splitting iff*
(a) $d(s, P_i) \leq d(s)/2$ for each $P_i \in \mathcal{P}$, and
(b) there is no C_4- or C_6-obstacle (defined below).

In our application we take $H = 2G$ with vertex set V, and form H^s by adding a vertex s and making it adjacent to all vertices. The partition $\mathcal{P}$ has the two copies of G as its sets. Let $k = \delta(G) + 1$. If a complete k-allowed splitting at vertex s is possible, then the result (ignoring s) is k-edge-connected. Further this raises the degree of each vertex by 1, and hence the split edges form a matching between the two copies of G, so that the resulting graph on V is the required permutation graph G^π.

Consider any subset X of V. Since every vertex of X has degree at least $k-1$, $d_H(X) \geq k - |X|$ and so $d_{H^s}(X) \geq k$. Thus V is k-expansive in H^s. Condition (a) trivially holds for our application. A C_6-obstacle by definition requires $\mathcal{P}$ to have at least 3 sets. So we have only to consider a C_4-obstacle. By definition this can exist only for k odd.

Bang-Jensen et al. define a C_4-obstacle as a partition $\{A_1, A_2, B_1, B_2\}$ of V with the following properties:
(i) $d_{H^s}(A_1) = d_{H^s}(A_2) = d_{H^s}(B_1) = d_{H^s}(B_2) = k$;
(ii) there is no edge between A_1 and A_2 nor between B_1 and B_2;
(iii) for some $P_i \in \mathcal{P}$, $N(s) \cap (A_1 \cup A_2) = N(s) \cap P_i$ and $d(s, P_i) = d(s)/2$.

Assume there is not a complete k-allowed splitting for our graph. Then there is a C_4-obstacle. Since in H^s the set V is k-expansive, properties (i) and (ii) imply that each of A_1, A_2, B_1, B_2 induces a connected subgraph (if two or more components, then each component has edge-boundary at least k). However, a component C in H has in our application $d_{H^s}(C) = |C|$, and so the only possibility to satisfy property (i) (recalling that $k = \delta(G) + 1$), is that each of A_1, A_2, B_1 and B_2 is a clique on k vertices. That is, $G = 2K_k$. This completes the proof of Theorem 16.

8 Open Questions

The three most obvious questions are:

- What is the complexity of finding the optimal augmentation for a given parameter?
- Is there a connectivity version for Theorem 16?
- Is there a diameter interpolation result?

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