A Note on S-Packing Colorings of Lattices

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Abstract

Let $a_1, a_2, \ldots, a_k$ be positive integers. An $(a_1, a_2, \ldots, a_k)$-packing coloring of a graph $G$ is a mapping from $V(G)$ to $\{1, 2, \ldots, k\}$ such that vertices with color $i$ have pairwise distance greater than $a_i$. In this paper, we study $(a_1, a_2, \ldots, a_k)$-packing colorings of several lattices including the infinite square, triangular, and hexagonal lattices. For $k$ small, we determine all $a_i$ such that these graphs have packing colorings. We also give some exact values and asymptotic bounds.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph. A set $X \subseteq V(G)$ is called an $i$-packing of $G$ if vertices of $X$ have pairwise distance greater than $i$. Let $a_1, \ldots, a_k$ be positive integers. An $(a_1, \ldots, a_k)$-packing coloring of a graph $G$ is a mapping $f$ from $V(G)$ to $\{1, 2, \ldots, k\}$ such that vertices with color $i$ have pairwise distance greater than $a_i$. For brevity, we will call this just an $(a_1, \ldots, a_k)$-coloring. For example, if all the $a_i$ are 1, this represents a normal $k$-coloring, and if all the $a_i$ are $r$, this represents a normal $k$-coloring of the $r$th power of $G$. Further, if $S = (a_1, a_2, \ldots)$ is a sequence of positive integers, we define the $S$-packing chromatic number $\chi_S(G)$ of $G$ as the smallest integer $k$ such that $G$ has an $(a_1, \ldots, a_k)$-coloring. If no such coloring of $G$ exists for any positive integer $k$, then we say $\chi_S(G) = \infty$.

The concept of $S$-packing chromatic number was mentioned in [8] and formally introduced and studied in [9]. There the focus was on the $S$-packing chromatic number of the infinite path $P_\infty$ and the computational complexity of $(a_1, a_2, a_3)$-colorability. A special case of this, called the packing chromatic number and written $\chi_p(G)$, is the case when $S = (1, 2, 3, \ldots)$. There have been several papers on the packing chromatic number, especially on the values for lattices and other infinite graphs [1, 2, 3, 6, 7, 8, 12, 13, 14], as well as on the computational complexity of the parameter [5, 8].

In this paper, we consider $\chi_S(G)$ of several lattices. The Cartesian product of graphs
and $H$, written $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ where $(u, v)$ and $(u', v')$ are adjacent if and only if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$. The two-way infinite path, written $P_\infty$, is the graph with the integer set $\mathbb{Z}$ as the vertex set, such that two vertices are adjacent if and only if they correspond to consecutive integers. The infinite square lattice, written $\mathbb{Z}^2$, is $P_\infty \Box P_\infty$. This graph has been the subject of several papers. The original [8] showed that $9 \leq \chi_\rho(\mathbb{Z}^2) \leq 23$; the lower bound was later improved to 10 by Fiala et al. [6] and then to 12 by Ekstein et al. [2], while the upper bound was improved by Holub and Soukal [13] to 17.

The infinite triangular lattice, denoted by $T$, is the graph obtained from $\mathbb{Z}^2$ by adding all edges of the form $\{(i, j), (i+1, j-1)\}$. The infinite hexagonal lattice, denoted by $\mathcal{H}$, is the 3-regular infinite plane graph where every face is a hexagon. Fiala et al. [6] showed that $\chi_\rho(\mathcal{H}) = 7$, while Finbow and Rall [7] showed that $\chi_\rho(T) = \infty$.

In this paper, we investigate $\chi_S(G)$ of the infinite one-row, square, two-row, triangular, and hexagonal lattices. We determine the cases where $\chi_S$ is small for these graphs. We also investigate the parameter for some special sequences, and give some exact values and asymptotic bounds.

2 Some Tools

2.1 Definitions

Throughout the paper, every sequence $S$ is a nondecreasing sequence of positive integers, and $a_i$ denotes the $i$th term of $S$. We define a sequence $(a_1, \ldots, a_k)$ as a minimal packing chromatic sequence (MPCS) for a graph $G$ if $G$ has an $(a_1, \ldots, a_k)$-coloring but does not have such a coloring if any of the $a_i$ are increased. Note that an MPCS might not exist: for example, a star has a $(1, a_2)$-coloring for all $a_2$. But MPCS exist for the lattices we consider.

A natural way to construct suitable colorings is to take an existing good coloring and partition one of the classes into several classes. We call such process splitting.

2.2 Density

We will use $X_i$ to denote an $i$-packing in the graph. Upper bounds on the size of $X_i$ are useful in establishing lower bounds on $\chi_S(G)$. Density was formally introduced by
Definition 1 [6] Let $G$ be a graph with finite maximum degree. Then the density of a set of vertices $X \subset V(G)$ is
\[
D_G(X) = \limsup_{r \to \infty} \max_{v \in V} \left\{ \frac{|X \cap N_r[v]|}{|N_r[v]|} \right\},
\]
where $N_r[v]$ is the set of vertices within distance $r$ of $v$.

Lemma 1 [6] If $(a_1, a_2, \ldots, a_k)$-coloring of $G$ has classes $X_{a_1}, X_{a_2}, \ldots, X_{a_k}$, then $\sum_{i=1}^{k} D_G(X_{a_i}) \geq 1$.

Lemma 2 [6] In the infinite square lattice $\mathbb{Z}^2$,
(a) $D_{\mathbb{Z}^2}(X_k)$ is at most $2/(k+1)^2$ if $k$ is odd, and at most $2/(k^2 + 2k + 2)$ if $k$ is even;
(b) $D_{\mathbb{Z}^2}(X_1 \cup X_2) \leq \frac{5}{8}$.

We will also need the bounds for the hexagonal lattice. The following lemma uses some ideas from [10].

Lemma 3 In the infinite hexagonal lattice $\mathcal{H}$,
(a) $D_{\mathcal{H}}(X_k) \leq 1/6m^2$ if $k = 4m - 1$;
(b) $D_{\mathcal{H}}(X_k) \leq 1/(6m^2 + 6m + 2)$ if $k = 4m + 1$; and
(c) $D_{\mathcal{H}}(X_k) \leq 2/(3m^2 + 3m + 2)$ if $k = 2m$.

Proof. (a) We decompose $\mathcal{H}$ into isomorphic subgraphs $F_m$ whose vertices are distance at most $4m - 1$ apart in $\mathcal{H}$. When $m = 1$, let $F_1$ be a 6-cycle. When $m > 1$, let $F_m$ include all the vertices within distance $2m - 2$ of a 6-cycle (including the 6-cycle itself). It is easy to see that $F_m$ has order $6m^2$ and diameter $2(2m - 2) + 3 = 4m - 1$, and so contains at most one vertex of a $(4m - 1)$-packing. The case of $F_2$ is illustrated in Figure 1a, where we use the dual graph $T$ to represent $\mathcal{H}$.

(b) We again decompose $\mathcal{H}$ into isomorphic subgraphs. When $m = 0$, let $G_0$ be a pair of adjacent vertices in $\mathcal{H}$. When $m > 0$, let $G_m$ include all the vertices within distance $2m$ of a pair of adjacent vertices (including the pair itself). The subgraph has order $2((2m+1)^2 - 2(m+1)m/2) = 6m^2 + 6m + 2$ and diameter $4m + 1$, and so contains at most one vertex of a $(4m + 1)$-packing. The case of $G_2$ is illustrated in Figure 1b.
(c) Let \( v \) be any vertex of \( \mathcal{H} \) and consider the set \( N_{m}[v] \) of vertices within distance \( m \) of \( v \). It can be checked that \( |N_{m}[v]| = \frac{(3m^2 + 3m + 2)}{2} \). Since the diameter of \( N_{m}[v] \) is \( 2m \), the set \( N_{m}[v] \) contains at most one vertex of a \( 2m \)-packing. Since \( \mathcal{H} \) is vertex-transitive, the density of a \( 2m \)-packing is thus at most \( 1/|N_{m}[v]| \). □

![Subgraphs containing only one vertex of a 7- or 9-packing in \( \mathcal{H} \) (shown in dual)](image)

Figure 1: Subgraphs containing only one vertex of a 7- or 9-packing in \( \mathcal{H} \) (shown in dual)

### 2.3 Computer Search

Several of the results in the paper were found by computer search. When a coloring is found, the coloring provides a certificate that the human can (in principle) check. However, in most of the cases, we do not know a human proof of the nonexistence of the coloring. All code was written in Java and run on a laptop. Most of the runs took at most a few minutes, though some executions for further results were aborted as taking too long. Source is provided at [http://people.cs.clemson.edu/~goddard/papers/latticeColorings](http://people.cs.clemson.edu/~goddard/papers/latticeColorings)

The basic building block is code PackingSearch that performs an exhaustive search for a coloring of a given (finite) graph \( G \) with given (finite) constraint sequence \( S \). That is, it tries all colors for the first vertex, all valid colors for the second vertex, all valid colors for the third vertex, and so on. (This can be sped up slightly if \( G \) is vertex-transitive, since then we can assume the first vertex has the first color.)

Most of the graphs in the paper are infinite. To show that an infinite graph \( H \) does not have a coloring with \( S \), we chose a finite induced subgraph \( G \) of \( H \) and ran PackingSearch to show that \( G \) does not have such a coloring.

To show that infinite \( H \) does have a coloring with \( S \), in some cases we got PackingSearch to print out a coloring of a subgraph \( G \) and by hand checked that the coloring
could be extended to a coloring of $H$. In the case of the one-row and two-row rectangular lattices, this process was automated by finding a coloring of a finite grid $G'$ with suitable “wraparound”: for example, if $H$ is the infinite path, then $G'$ is a cycle of suitable length. Note that, in lifting the coloring of $G'$ to $H$, one has to check that it remains a valid coloring; in particular that the pre-images of any vertex in $G'$ are sufficiently far apart.

We also used code to try to automate finding MPCS. This was achieved by an exhaustive search on the sequences $S$. That is, try all possible values of $a_1$, try all values of $a_2$ with $a_2 \geq a_1$ and so on. This process becomes finite since, if there does not exist a coloring for the sequence $(a_1, \ldots, a_k)$, then there does not exist a coloring with any of these values increased. Further, processing time is reduced by noting that if there is a coloring for $(a_1, \ldots, a_k)$, then there is a coloring for all sequences that start with those values. For example, our square and hexagonal lattices are bipartite; thus once the $(1,1)$-coloring is found, all other minimal sequences have $a_2 \geq 2$.

3 The Path

The $S$-chromatic number of the infinite path $P_\infty$ was the focus of [9]. We consider here the existence of packing colorings with few colors.

A natural way to color the path $P_\infty$ is to start with all vertices with the same color. Then repeatedly split the vertices of some color into some number of colors. If vertices of color $i$ were at least $a_i + 1$ apart before and are split into $r$ colors, then two vertices of the same new color are at least $r(a_i + 1)$ apart. One can also view this process as it changes the sequence $S$. By splitting $a_i$ into $r$ parts for some integer $r$, we mean the operation of replacing $a_i$ with $r$ copies of $r(a_i + 1) - 1$, and sorting the resultant sequence. For example, if we have $(1,3,3)$ and split $a_1$ into 3 parts, we obtain the sequence $(3,3,5,5,5)$. We define a splintered sequence as one that can be obtained from the sequence $(0)$ by some sequence of splits.

For example, there are 17 splintered sequences of length at most 5:

(1,1), (2,2,2), (1,3,3), (3,3,3,3), (2,2,5,5), (1,5,5,5), (1,3,7,7), (4,4,4,4,4),
(3,3,5,5,5), (3,3,3,7,7), (2,5,5,5,5), (2,2,8,8,8), (2,2,5,11,11), (1,7,7,7,7), (1,5,5,11,11),
(1,3,11,11,11), and (1,3,7,15,15).
Proposition 1  Every splintered sequence is an MPCS for the infinite path.

Proof.  By induction, $\sum_{i=1}^{k} 1/(a_i + 1) = 1$ for any splintered sequence. By Lemma 1, it follows that every splintered sequence is a MPCS. □

Note that each splintered sequence corresponds to a partition of $\mathbb{Z}$ into arithmetic progressions. This is called an exact cover or disjoint covering system of $\mathbb{Z}$. Indeed, a splintered sequence corresponds to what is called a natural exact cover [11].

For a small number of colors, we were able to determine all MPCS:

Proposition 2  For the infinite path $P_\infty$ and $k \leq 5$, all minimal packing chromatic sequences are splintered sequences except for $(2, 4, 4, 6)$, $(2, 3, 4, 4, 9)$, $(2, 3, 3, 8, 8)$, and $(2, 3, 3, 4, 12)$.

Proof.  We used a computer to determine which sequences are colorable and which are not. From the set of colorable sequences, one can then read off those that are minimal. There are 21 of these, of which 17 are the splintered sequences, given above.

The remaining four sequences do not have $\sum_i 1/(a_i + 1) = 1$, and hence are not splintered. For these, a coloring with minimum period is given by:

- $(2, 4, 4, 6)$: 1.2.3.1.4.5.2.1.3.4.1.2.5.3.1.4.2.1.3.5.4
- $(2, 3, 4, 4, 9)$: 1.2.3.1.4.2.1.3.5.4
- $(2, 3, 3, 8, 8)$: 1.2.3.1.4.2.1.3.5
- $(2, 3, 3, 4, 12)$: 1.2.3.1.4.2.1.3.5.1.2.3.4

□

The problem for larger $k$ seems more complicated.

4  The Infinite Square Lattice

4.1  Few Colors

We start with the question of the existence of packing colorings with at most six colors.

Proposition 3  $\chi_S(\mathbb{Z}^2) = 2$ if and only if $a_1 = a_2 = 1$. There is no sequence $S$ such that $\chi_S(\mathbb{Z}^2) = 3$ or $\chi_S(\mathbb{Z}^2) = 4$. 

Proof. The first statement is trivial. So assume \( a_2 \geq 2 \), and consider some vertex \( v \) with color 1. Its four neighbors have to be assigned with distinct colors, and so at least five colors are needed. \( \square \)

**Proposition 4** \( \chi_S(\mathbb{Z}^2) = 5 \) if and only if either

(i) \( a_1 = 1, a_2 \geq 2, \) and \( a_5 \leq 3, \) or

(ii) \( a_1 = a_5 = 2. \)

Proof. To show the “if” part, it suffices to construct a \((1, 1, 3, 3, 3, 3)\)- and a \((2, 2, 2, 2, 2, 2)\)-coloring of \( \mathbb{Z}^2 \). We show the patterns in Figure 2. In the illustration, each vertex of \( \mathbb{Z}^2 \) is represented by a square, and the coloring repeats in all directions.

![Colorings of \( \mathbb{Z}^2 \)](https://example.com/figure2.jpg)

On the other hand, assume that \( \chi_S(\mathbb{Z}^2) = 5 \). By using the density argument of Lemmas 1 and 2, it suffices to show that there is no \((1, 1, 2, 2, 3, 3)\)-coloring of \( \mathbb{Z}^2 \).

Suppose there exists \((1, 2, 2, 2, 4, 4)\)-coloring of \( \mathbb{Z}^2 \) and consider a vertex of color 1 and the 5 \( \times \) 5 portion of the lattice around it (that is, color 1 is at position \([3, 3]\)). By symmetry, we may assume that color 2 is at \([2, 3]\), color 3 is at \([3, 2]\), color 4 is at \([3, 4]\), and color 5 is at \([4, 3]\). If \([2, 4]\) is colored with 1, then one of \([1, 4]\) and \([2, 5]\) has to be colored with 5, but this cannot be done by distance constraint. So \([2, 4]\) has to be colored with 3. Similarly \([2, 2]\) has to be colored with 4. Then \([1, 3]\) has to be colored with 1, but we cannot color \([1, 4]\). \( \square \)

**Proposition 5** \( \chi_S(\mathbb{Z}^2) = 6 \) if and only if \((a_1, a_2, a_3, a_4, a_5, a_6)\) is \((2, 2, 2, 2, 3, 3)\) or \((1, 2, 2, 2, 4, 4)\).

Proof. To show the “only if” part, it suffices to show that four colorings do not exist. By computer search it follows that there is no \((2, 2, 2, 3, 3, 3)\)-, \((2, 2, 2, 3, 4)\)-, \((1, 2, 2, 3, 2, 4)\)- nor \((1, 2, 2, 2, 4, 5)\)-coloring of the 7 \( \times \) 7 grid.
To show that “if” part, we need two constructions. First, a \((2, 2, 2, 2, 3, 3)\)-coloring can be obtained from the \((2, 2, 2, 2, 2)\)-coloring of Figure 2b by splitting color 5 into two colors. This is shown in Figure 3a. There every row is tiled with the tile shown, but the tiling is shifted 3 cells left in the row above and shifted 3 cells right in the row below.

Second, a \((1, 2, 2, 2, 4, 4)\)-coloring is illustrated in Figure 3b. The rows of the square lattice are partitioned into consecutive pairs. Every pair of rows is tiled with the tile shown, but the tiling is shifted 4 cells left in the pair above and shifted 4 cells right in the pair below. □

\[
\begin{array}{cccccccc}
1 & 4 & 5 & 2 & 3 & 1 & 4 & 6 \\
6 & 4 & 1 & 2 & 3 & 1 & 4 & 2 \\
\end{array}
\]

(a) \((2, 2, 2, 2, 3, 3)\)

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 1 & 4 & 5 & 2 & 1 \\
6 & 4 & 1 & 2 & 3 & 1 & 4 & 6 \\
\end{array}
\]

(b) \((1, 2, 2, 2, 4, 4)\)

Figure 3: Colorings of \(\mathbb{Z}^2\)

4.2 More Colors

When all the \(a_i\) are the same, we get what some have called the \(k\)-distance chromatic number. Fertin et al. [4] determined the value for the square lattice:

**Proposition 6** [4] Let \(S = (k, k, k, \ldots)\) where \(k\) is a positive integer, then

\[
\chi_S(\mathbb{Z}^2) = \begin{cases} 
\frac{(k+1)^2}{2}, & \text{if } k \text{ is odd;} \\
\frac{k^2 + 2k + 2}{2}, & \text{if } k \text{ is even.}
\end{cases}
\]

We next investigate the case that \(S\) is an arithmetic progression. It is known that \(12 \leq \chi_S(\mathbb{Z}^2) \leq 17\) [2, 13] when \(S = (1, 2, 3, \ldots)\). In contrast, we will show that \(\chi_S(\mathbb{Z}^2) = \infty\) if \(S\) is any nonconstant arithmetic progression other than \((1, 2, 3, \ldots)\).
Proposition 7 \( \chi_S(\mathbb{Z}^2) = \infty \) if \( S \) is a nonconstant arithmetic progression other than \((1, 2, 3, \ldots)\).

Proof. It suffices to prove (a) that \( \chi_S(\mathbb{Z}^2) = \infty \) if \( S = (1, 3, 5, \ldots) \) and (b) that \( \chi_S(\mathbb{Z}^2) = \infty \) if \( S = (2, 3, 4, \ldots) \).

By Lemmas 1 and 2, since
\[
\sum_{i=1}^{\infty} D_{\mathbb{Z}^2}(X_{2i-1}) \leq \sum_{i=1}^{\infty} \frac{2}{(2i)^2} = \frac{\pi^2}{12} < 1,
\]
we have (a). Further, since
\[
\sum_{i=2}^{\infty} D_{\mathbb{Z}^2}(X_i) \leq \sum_{k=1}^{\infty} \frac{2}{(2k + 2)^2} + \sum_{k=1}^{\infty} \frac{2}{4k^2 + 4k + 2} = \frac{\pi^2 - 6}{12} + \frac{\pi \tanh(\pi/2) - 2}{2} < 1,
\]
we have (b). \( \square \)

We have seen that \( \chi_S(\mathbb{Z}^2) = \infty \) if \( S = (2, 3, 4, 5, \ldots) \). This raises a question: What if we prepend some 2’s to the sequence? We show that if we prepend only one 2, the chromatic number remains infinite, and if we prepend three 2s, the chromatic number becomes finite.

Proposition 8 (a) If \( S = (2, 2, 3, 4, 5, \ldots) \), then \( \chi_S(\mathbb{Z}^2) = \infty \).
(b) If \( S = (2, 2, 2, 3, 4, 5, \ldots) \), then \( \chi_S(\mathbb{Z}^2) = 7 \).

Proof. (a) Since
\[
D_{\mathbb{Z}^2}(X_2) + \sum_{i=2}^{\infty} D_{\mathbb{Z}^2}(X_i) \leq \frac{1}{5} + \sum_{k=1}^{\infty} \frac{2}{(2k + 2)^2} + \sum_{k=1}^{\infty} \frac{2}{4k^2 + 4k + 2}
= \frac{1}{5} + \frac{\pi^2 - 6}{12} + \frac{\pi \tanh(\pi/2) - 2}{2}
= 0.96313 < 1,
\]
\( \chi_S(\mathbb{Z}^2) = \infty \).

(b) Proposition 5 tells us that \( \chi_S(\mathbb{Z}^2) \geq 7 \). For the upper bound, we present the pattern of a \((2, 2, 2, 2, 3, 4, 5)\)-coloring (obtained by splitting the \((2, 2, 2, 2)\)-coloring) in Figure 4. \( \square \)

We do not know whether \( \chi_S(\mathbb{Z}^2) \) is finite for the sequence \( S = (2, 2, 3, 4, 5, \ldots) \). That is, density arguments do not preclude it and computer search takes too long.
5 The Infinite Two-Row Lattice

The infinite two-row lattice, written $G_{2,\infty}$, is $P_2 \square P_\infty$. We were able to determine the sequences for which $\chi_S(G_{2,\infty}) \leq 5$.

**Proposition 9** For $k \leq 5$, the MPCS for $G_{2,\infty}$ are $(1,1)$, $(2,2,2)$, $(1,3,3)$, $(2,2,3,3)$, and $(1,3,3,5,5)$.

**Proof.** As before, the impossibility results are generated by computer. Illustration of the actual colorings are given in Figure 5. \[\square\]

![Colorings of $G_{2,\infty}$](image)

The problem seems more complicated for larger $k$.

We also study the case that $S$ is an arithmetic progression. Proposition 9 says that $\chi_S(G_{2,\infty}) = 5$ if $S = (1,2,3,4,\ldots)$ (also see [8]). To obtain a general result for arithmetic progressions with common difference 1, we need an idea that appeared in [8]:

**Proposition 10** [8, 9] Let $S = (a, a+1, a+2, a+3,\ldots)$. Then $(e-1)a \leq \chi_S(P_\infty) \leq 2a + 3$. 

10
Proposition 11  Let $S = (a, a+1, a+2, a+3, \ldots)$. Then $(e^2 - 1)(a-1) \leq \chi_S(G_{2,\infty}) \leq 8a + 12$.

Proof. Suppose that $\chi_S(G_{2,\infty}) = k$. It is easy to see that the density of a $k$-packing of $G_{2,\infty}$ is at most $1/(2k)$. Thus

$$2 \leq \sum_{i=1}^{k} \frac{1}{a + (i-1)} \leq \int_{a-1}^{a+k-1} \frac{1}{x} \, dx = \ln(a + k - 1) - \ln(a - 1).$$

The lower bound follows by solving for $k$.

The upper bound comes from the following construction: We first color the first row of $G_{2,\infty}$ with colors $1, 2, \ldots, 2a + 3$. This can be done by Proposition 10. We then color the second row of $G_{2,\infty}$ with the remaining colors. Note that the distance constraint of color $2a + 4$ is $3a + 3$, and by Proposition 10, $\chi_S(P_\infty) \leq 6a + 9$ if $S = (3a + 3, 3a + 4, 3a + 5, \ldots)$. Thus we can color the second row of $G_{2,\infty}$ with no more than $6a + 9$ colors. This gives a coloring that uses no more than $8a + 12$ colors. \(\square\)

Remark: The above idea shows that for any arithmetic progression $S$ we have $\chi_S(P_\infty \Box G) < \infty$ for any finite graph $G$: we can use disjoint colors to color each $P_\infty$ fiber.

The next proposition improves Proposition 11 for the particular case that $S = (2, 3, 4, \ldots)$.

Proposition 12 $10 \leq \chi_S(G_{2,\infty}) \leq 14$ if $S = (2, 3, 4, \ldots)$.

Proof. The lower bound is obtained directly from the density. To prove the other direction, we present a coloring pattern in Figure 6. \(\square\)

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Figure 6: A $(2, 3, 4, \ldots, 15)$-coloring of $G_{2,\infty}$

6 The Infinite Triangular Lattice

Finbow and Rall [7] showed that $\chi_S(T) = \infty$ if $S = (1, 2, 3, \ldots)$. Hence, we know that $\chi_S(T) = \infty$ if $S$ is strictly increasing.
We were able to determine the sequences for which \( \chi_S(T) \leq 6 \).

Proposition 13. For the triangular lattice \( T \) and \( k \leq 6 \), the MPCS are \((1,1,1)\), \((1,1,2,2,2)\) and \((1,1,3,3,3,3)\).

Proof. The proof that stricter sequences are not possible is by computer.

The \((1,1,2,2,2)\)-coloring is obtained from the proper 3-coloring by splitting one color class into three colors. The \((1,1,3,3,3,3)\)-coloring is obtained from the proper 3-coloring by splitting one color class into four colors. We illustrate these in Figure 7 by using the fact that \( T \) is the dual of the hexagonal lattice \( H \). □

\[ \begin{array}{ccc}
  \begin{array}{ccc}
    2 & 1 & 3 \\
    1 & 4 & 2 \\
    1 & 3 & 2 \\
    1 & 5 & 2
  \end{array} & \begin{array}{ccc}
    2 & 1 & 3 \\
    1 & 4 & 2 \\
    1 & 5 & 2 \\
    1 & 6 & 2
  \end{array} & \begin{array}{ccc}
    2 & 1 & 3 \\
    1 & 4 & 2 \\
    1 & 5 & 2 \\
    1 & 6 & 2
  \end{array}
\end{array} \]

Figure 7: Colorings of the triangular lattice \( T \) (shown in dual)

7 The Infinite Hexagonal Lattice

Proposition 14. For the hexagonal lattice \( H \) and \( k \leq 5 \), the MPCS are \((1,1)\), \((2,2,2,2)\), \((1,3,3,3)\), and \((2,2,2,3,3)\).

Proof. The proof that stricter sequences are not possible is by computer.

The hexagonal lattice \( H \) is bipartite. The square of \( H \) is 4-colorable by a greedy strategy (or see Proposition 15b below). The \((1,3,3,3)\)-coloring is obtained by taking the \((1,1)\)-coloring and splitting one color class into three. The \((2,2,2,3,3)\)-coloring is obtained from the \((2,2,2,2)\)-coloring and splitting one color class into two. See Figure 8, where we have used the fact that \( H \) is dual to \( T \). □
Jacko and Jendrol’ [10] studied the $k$-distance chromatic number for the hexagonal lattice:

**Proposition 15** [10] Let $S = (k, k, k, \ldots)$ where $k$ is a positive integer, then

(a) $\chi_S(H) = \lceil \frac{3}{8} (k + 1)^2 \rceil$ if $k$ is odd;
(b) $\chi_S(H) = 4$ if $k = 2$, $\chi_S(H) = 11$ if $k = 4$, and $\chi_S(H) = 20$ if $k = 6$;
(c) $\frac{3}{2} k^2 + \frac{3}{4} k + 2 \leq \chi_S(H) \leq \lceil \frac{3}{8} (k + \frac{4}{3})^2 \rceil$ if $k$ is even and $k \geq 8$;

where $[x]$ is the nearest integer to $x$.

Finally, we investigate the case that $S$ is an arithmetic progression. It is known that $\chi_S(H) = 7$ if $S = (1, 2, 3, 4, \ldots)$ [6]. We show that $\chi_S(H) = \infty$ if $S$ is a nonconstant arithmetic progression other than $(1, 2, 3, 4, \ldots)$.

**Proposition 16** $\chi_S(H) = \infty$ if $S$ is a nonconstant arithmetic progression other than $(1, 2, 3, \ldots)$.

**Proof.** It suffices to prove that (a) $\chi_S(H) = \infty$ if $S = (1, 3, 5, \ldots)$ and (b) $\chi_S(H) = \infty$ if $S = (2, 3, 4, \ldots)$.

By Lemma 3, since

$$
\sum_{i=0}^{\infty} D_H(X_{4i+1}) + \sum_{i=1}^{\infty} D_H(X_{4i-1}) \leq \sum_{i=0}^{\infty} \frac{1}{6i^2 + 6i + 2} + \sum_{i=1}^{\infty} \frac{1}{6i^2} = 0.92680 < 1,
$$

we have (a). Further, since

$$
\sum_{i=2}^{\infty} D_H(X_i) = \sum_{i=1}^{\infty} \left( \frac{2}{3i^2 + 3i + 2} + \frac{1}{6i^2 + 6i + 2} + \frac{1}{6i^2} \right) = 0.99386 < 1,
$$

we have (b). □
8 Conclusion

We conclude this paper by listing a few open problems:

1. Determine the sequences $S$ for which $\chi_S(\mathbb{Z}^2)$ is finite. In particular, is it finite for $S = (2, 2, 2, 3, 4, 5, \ldots)$?

2. What is the exact value of $\chi_S(G_{2,\infty})$ for $S = (2, 3, 4, \ldots)$?

3. While it was not the focus of this paper, we think the open problem of determining $\chi_S(\mathbb{Z}^2)$ for the sequence $S = (1, 2, 3, 4, \ldots)$ remains of interest.

References


