Independent Domination, Colorings and the Fractional Idomatic Number of a Graph

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Abstract

A set $S$ of vertices is an independent dominating set if it is both independent and dominating, and the idomatic number is the maximum number of vertex-disjoint independent dominating sets. In this paper we consider a fractional version of this. Namely, we define the fractional idomatic number as the maximum ratio $|\mathcal{F}|/m(\mathcal{F})$ over all families $\mathcal{F}$ of independent dominating sets, where $m(\mathcal{F})$ denotes the maximum number of times an element appears in $\mathcal{F}$. We start with some bounds including a connection with dynamic colorings. Then we show that the independent domination number of a planar graph with minimum degree 2 is at most half its order, and its fractional idomatic number is at least 2. Moreover, we show that an outerplanar graph of minimum degree 2 has idomatic number at least 2. We conclude by providing formulas for the parameters for the join, disjoint union and lexicographic product of graphs, while providing some bounds for cubic graphs.

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1 Introduction

A set \( S \) of vertices in a graph is an **independent dominating set** if it is both independent and dominating. That is, no two vertices of \( S \) are adjacent, and every other vertex is adjacent to at least one vertex of \( S \). Equivalently, \( S \) is a maximal independent set. The **idomatic number** of a graph \( G \) is defined as the maximum number of vertex-disjoint independent dominating sets of \( G \). We will denote this by \( \text{idom}(G) \). This terminology was introduced by Zelinka [17], but the parameter was originally defined by Cockayne and Hedetniemi [5]. In this paper we consider a fractional version of the idomatic number of a graph.

For a family \( \mathcal{F} \) of sets, we define \( m(\mathcal{F}) \) as the maximum number of times an element appears in \( \mathcal{F} \). (In hypergraph terminology this is the maximum degree of \( \mathcal{F} \).) Then we define the **fractional idomatic number** of graph \( G \), denoted \( \text{FIDOM}(G) \), as

\[
\text{FIDOM}(G) = \max \frac{|\mathcal{F}|}{m(\mathcal{F})}
\]

where the maximum is taken over all families \( \mathcal{F} \) of independent dominating sets of \( G \). We note that this parameter should be defined as the supremum, but by standard fractional (linear programming) arguments (see for example [16]), one can show that the supremum is achieved.

The independent domination number \( i(G) \) of graph \( G \) is the minimum size of an independent dominating set. Note that \( \text{FIDOM}(G) \geq \text{idom}(G) \). On the other hand,

\[
\text{FIDOM}(G) \leq n/i(G)
\]

for a graph \( G \) of order \( n \), since \( \sum_{F \in \mathcal{F}} |F| \leq nm(\mathcal{F}) \) but \( \sum_{F \in \mathcal{F}} |F| \geq i(G) |\mathcal{F}| \).

For some background on the idomatic number see [17, 5, 15]; for more information on independent dominating sets see [9]; for the fractional version of domatic number see [1]; and for the fractional version of total domatic number (related to coupon coloring or thoroughly dispersed colorings) see [10].

We proceed as follows. In Section 2 we provide some preliminary observations and bounds. In Section 3 we discuss colorings and show a connection between dynamic (partial) colorings and the fractional idomatic number. In Section 4 we consider planar graphs, and show that the independent domination number of a
planar graph with minimum degree 2 is at most half its order, and its fractional
domatic number is at least 2. Though such a planar graph need not have two
disjoint independent dominating sets, we show this is true for an outerplanar graph of
minimum degree 2. Finally in Section 5 we consider some graph families, and provide
formulas for the join, disjoint union and lexicographic product, while providing some
bounds for cubic graphs.

2 Preliminaries

As a warm-up, consider a nontrivial tree $T$. Then $\text{FIDOM}(T) = 2$. This can be
seen by noting that for a leaf $v$, every independent dominating set contains either
$v$ or its neighbor. So $\text{FIDOM}(T) \leq 2$. On the other hand, both color classes are
independent dominating sets, and so $\text{FIDOM}(T) \geq \text{idom}(T) \geq 2$.

The above idea can be generalized in several ways. The lower bound from bi-
partiteness will be extended in the next section. The upper bound argument imme-
diately generalizes to:

**Lemma 1** If graph $G$ has minimum degree $\delta$, then $\text{FIDOM}(G) \leq \delta + 1$.

Examples of equality in Lemma 1 are the cycles of length a multiple of 3. More
generally, cycles have $\text{FIDOM}(C_n) = n/\lceil n/3 \rceil$, since a suitable family can be ob-
tained by taking a minimum independent dominating set $S$ along with all rotations
of $S$.

Now, recall that unlike for example the domination number, the independent
domination number of a graph of order $n$ can be as much as $n - o(n)$, even with
prescribed minimum degree. (For more precise result, see Favaron [6].) For such a
t Graph $G$, we have $\text{idom}(G) = 1$ and $\text{FIDOM}(G) \leq 1 + o(1)$. (Note that $\text{FIDOM}(G) = 1$ only if the graph $G$ has an isolate.)

Even for graph families where there are better bounds on the independent dom-
ination number, one can still see this behavior. Consider for example a claw-free
t graph $G$ of order $n$. Allen and Laskar [3] showed that such $G$ has independent dom-
inature number equal to its domination number, and thus $i(G) \leq n/2$ if no isolates.
However, one can construct such a graph $G$ with arbitrarily large minimum degree
that has $\text{FIDOM}(G) = 1 + o(1)$ as follows. Start with a clique $X$ on $da$ vertices and partition $X$ into disjoint sets $S_1, \ldots, S_a$ each of size $d$. Then for every $S_i$, add a vertex $y_i$ whose neighborhood is $S_i$. The resultant split graph $G_{a,d}$ is claw-free and has minimum degree $d$. Further, every independent dominating set uses at least $a - 1$ vertices of $\{y_1, \ldots, y_a\}$. Thus $\text{FIDOM}(G_{a,d}) \leq a/(a - 1)$ (actually equality occurs).

3 Colorings

MacGillivray and Seyfarth [13] provided an upper bound for the independent domination number in terms of the chromatic number.

**Theorem 2** [13] If $G$ is a connected graph on $n$ vertices with chromatic number $k \geq 3$, then $i(G) \leq (k - 1)n/k - (k - 2)$.

We note that actually one does not need the coloring to use all the vertices. Let us define $\chi_P(G)$ as the minimum number of colors needed to properly color some of the vertices of the graph $G$ such that every vertex has at least one colored neighbor. Equivalently, this is the minimum chromatic number of (the subgraph induced by) a total dominating set. See for example, [15] or [4] (where they use the notation $\gamma$). By a similar proof to the above theorem (see for example [9]), it follows that:

**Lemma 3** If $G$ is a connected graph on $n$ vertices with $\chi_P(G) = k \geq 3$, then $i(G) \leq (k - 1)n/k - (k - 2)$.

A different extension of Theorem 2 is to consider dynamic coloring. An $r$-dynamic coloring, also called $r$-hued coloring and other names, is a proper coloring of the vertices such that every vertex $v$ has at least $\min\{d(v), r\}$ colors in its neighborhood, where $d(v)$ is the degree of vertex $v$. See for example Jahanbekam et al. [11].

**Lemma 4** If graph $G$ with minimum degree at least $r$ has an $r$-dynamic coloring using $k$ colors, then $\text{FIDOM}(G) \geq k/(k - r)$ and therefore $i \leq (k - r)n/k$. 

4
Proof. For each color $c$, create a maximal independent set by starting with all vertices of color $c$. In this way we obtain $k$ independent dominating sets; call the resultant family $\mathcal{F}$. Since each vertex has at least $r$ colors in its neighborhood, it follows that $m(\mathcal{F}) \leq k - r$. \qed

In particular:

**Theorem 5** For graph $G$ with chromatic number $k$ and no isolated vertex, it holds that $\text{FIDOM}(G) \geq k/(k - 1)$.

We note that one can get equality in the above results even with non-optimal colorings. For example, let $K_m^{(s)}$ denote the complete $s$-partite graph with $m$ vertices in each partite set and let $r \in \{1, \ldots, m\}$. Then one can pick $r$ vertices from each partite set and give all $rs$ vertices distinct colors, and extend to a proper coloring. Then the result is an $r$-dynamic coloring with $rs$ colors so that by Lemma 4, it follows that $\text{FIDOM}(K_m^{(s)}) \geq s/(s - 1)$, which is the correct value.

We can say a little more about equality in the above results. Say $G^*$ is a graph with equality in Lemma 4. This requires that when one forms the family $\mathcal{F}$, every vertex is in exactly $k - r$ sets. Thus every vertex sees exactly $r$ colors. Further, for every color, the vertices not adjacent to that color form an independent set in $G^*$. For each vertex $v$, let $M_v$ be the colors not in its neighborhood (where we include $v$’s own color in $M_v$). Note that $|M_v| = k - r$. Consider adjacent vertices $v$ and $w$. By definition, the color of $v$ is not in $M_w$ and vice versa. Further, we noted above that there is no edge between vertices not seeing any particular color. That is, the sets $M_v$ and $M_w$ are disjoint. In particular, it follows that the graph $G^*$ is homomorphic to the Kneser graph $KG_{k,k-r}$.

For example, if $r = k/2$, it follows that to get equality the graph $G^*$ must be bipartite. One example is $K_{r,r}$. But one can also take the path $P_4$ and expand each vertex to an independent set of size $r$ (that is, the lexicographic product $P_4[rK_1]$). Another example of equality for the $k = 4$ and $r = 2$ case is shown below in Figure 2.
4 Planar Graphs

We now consider planar graphs. At one extreme, one can ask for the maximum idomatic or fractional idomatic number of a planar graph. Since the minimum degree is at most 5, by Lemma 1 the largest both parameters can attain is 6. An example of a planar graph achieving 6 is the icosahedron; this has six disjoint independent dominating sets of size 2, each consisting of a vertex and the unique vertex at distance 3 from it.

So for the rest of the section we look at lower bounds. As a consequence of Theorems 2 and 5 and the Four Color Theorem, it follows that:

**Theorem 6** Let $G$ be a planar connected graph on $n$ vertices.

(a) If $n \geq 2$, then $\text{FIDOM}(G) \geq 4/3$.
(b) If $n \geq 10$, then $i(G) \leq 3n/4 - 2$.

(Note that in the survey [9] it was incorrectly stated that the bound in (b) holds for all $n$. ) These two bounds are sharp because of the corona $K_4$ with feet:

![Figure 1. A planar graph $G$ with $\text{FIDOM}(G) = 4/3$](image)

4.1 Minimum degree 2

But what happens if one considers planar graphs of minimum degree at least 2? It turns out that the maximum independent domination of such a graph is half its order, as we now show. The key is the result of Kim, Lee and Park [12]:

**Theorem 7** [12] Every connected planar graph has a 2-dynamic coloring using at most 4 colors, except for $C_5$.

Since the 5-cycle has $\text{FIDOM}(C_5) = 5/2$, it follows from Theorem 7 and Lemma 4 that:
**Theorem 8** For any planar graph $G$ with minimum degree at least 2, it holds that $\text{FIDOM}(G) \geq 2$ and $i(G) \leq n/2$.

The independent domination number bound is sharp because of the 4-cycle. Consider also the following construction. Start with the 4-cycle $v_1v_2v_3v_4v_1$ and take $s \geq 2$. Then for $i = 1, 2$, introduce $s$ new vertices whose neighbors are the pair $\{v_i, v_{i+2}\}$. Call the resulting graph $H_s$. Since if one takes $v_1$, one can take neither $v_2$ nor $v_4$, it follows that $i(H_s) = \min\{s + 2, 2s\}$, which equals $n/2$ since $s \geq 2$. The graph $H_4$ is illustrated below.

![Figure 2. The planar graph $H_4$ with $i(H_4) = n/2$](image)

It is unclear if these are the only cases of $i(G) = n/2$. But the graphs $G$ that achieve $\text{FIDOM}(G) = 2$ are more numerous. For example, it is sufficient that the graph contain a 4-cycle with two adjacent vertices $x$ and $y$ that have degree 2 in the original graph. Every independent dominating set contains exactly one of $x$ and $y$.

Here is a picture:

![Figure 3. A planar graph with FIDOM = 2 (but $i < n/2$)](image)

Also, note that there are planar graphs of minimum degree 2 that do not have two disjoint independent dominating sets. One construction is to start with $K_4$
with vertex set \( \{v_1, \ldots, v_4\} \) and \( s \geq 1 \). Then for each pair \( \{v_i, v_j\} \) introduce \( s \) new vertices whose neighbors are the pair \( \{v_i, v_j\} \). Call the resulting graph \( R_s \). The graph \( R_1 \) is drawn in Figure 4. Since one can take only 1 vertex from the \( K_4 \), it follows that every pair of independent dominating sets overlap in at least \( s \) vertices. Further, \( i(R_s) = 3s + 1 \) while the order is \( 6s + 4 \).

![Figure 4. The planar graph \( R_1 \) with idom\((R_1) = 1 \)](image)

### 4.2 Higher minimum degree

What about planar graphs with minimum degree 3? The 5-prism \( G = C_5 \square K_2 \) has independent domination number 4. Computer search suggests this might be the worst possible. That is, maybe every planar graph \( G \) of order \( n \) with minimum degree at least 3 has \( i(G) \leq 2n/5 \). This would follow for example if it was true that every such graph has a 3-dynamic coloring using at most 5 colors. An example of a graph that has such a coloring is one that contains a perfect dominating set \( S \) (that is, \( S \) is a packing whose closed neighborhoods cover all the vertices). For, by Theorem 7, one can color \( G - S \) using at most 4 colors so that every vertex sees at least two colors (provided \( G - S \) is not \( C_5 \)), and then one can extend to a 3-dynamic coloring of \( G \) using at most 5 colors by assigning to all vertices of \( S \) the same new color.

Asymptotically, the best construction we know produces examples with independent domination number \( 4/11 \) their order. Here it is. Let \( H \) be a planar graph of minimum degree at least 2. Then, for each vertex \( v \) of \( H \), add a vertex-disjoint copy of the 5-prism and join \( v \) to exactly one vertex of the 5-prism. Call the resulting graph \( P(H) \). Then \( P(H) \) has minimum degree 3 and order \( n = 11|V(H)| \). The graph \( P(K_3) \) is drawn below. Call the vertex \( v \) together with its associated copy of
a 5-prism a “unit” (of order 11). Since every independent dominating set of $P(H)$ contains at least four vertices from every unit, it follows that $i(P(H)) = 4n/11$.

![Figure 5. The planar graph $P(K_3)$](image)

Turning to triangulations, Matheson and Tarjan [14] showed that every planar triangulation has three disjoint dominating sets. Maybe:

**Question 1** Does every planar triangulation have three disjoint independent dominating sets?

This would imply that every planar triangulation $T$ of order $n$ has $i(T) \leq n/3$. We know of only two examples where the independent domination number is one-third the order, namely the triangle $K_3$ and the octahedron $K_{2,2,2}$. For large graphs, the largest independent domination number asymptotically we know has independent domination about $5/19$ its order.

Start with some triangulation $T$ on $3m$ vertices containing $m$ disjoint triangles, say $D$. In each of the pairwise-disjoint faces of $D$, add a single vertex. In each of the other faces, add a triangle and join each vertex of the triangle to two vertices on the faces. Call the resulting graph $Z(T,D)$. An example is shown below. (This is similar to the $U(G)$ construction in [10] or the o-join construction in Finbow et al. [7].)
Now, the order of $Z(T, D)$ is $3m + m + (5m - 4) \cdot 3 = 19m - 12$. Every independent dominating set of $Z(T, D)$ requires at least one vertex from the added triangles; and at least one vertex to dominate the added singletons. So $i(Z(T, D)) \geq 6m - 4$.

4.3 Outerplanar graphs

Maximal outerplanar graphs are straight-forward. They are 3-colorable and every color class is an independent dominating set. Since its minimum degree is 2, it follows that $\text{idom}(G) = \text{FIDOM}(G) = 3$ for a maximal outerplanar graph $G$. Also $i(G) \leq n/3$ for order $n$. Further, one can obtain a graph with independent domination number $n/3$ by starting with disjoint triangles, and adding edges while maintaining one vertex of degree 2 in each triangle. See Figure 7. More generally, the same idea works for $k$-trees.
But what happens in a general outerplanar graph $G$? Since it is 3-chromatic, Theorem 5 shows that $\text{FIDOM}(G) \geq 3/2$ and that $i(G) \leq 2n/3$. These values are achieved by the corona $K_3$ with feet.

But that suggests imposing a minimum-degree or connectivity condition. We noted earlier that there are outerplanar graphs $G$ with minimum degree 2 and $\text{FIDOM}(G) = 2$. We show next that an outerplanar graph with minimum degree at least 2 has two disjoint independent dominating sets.

We prove first a slightly stronger statement for a special subset of these graphs. Define a \textit{restrictive pair} as a pair of adjacent vertices of degree 2 that lie in a 4-cycle. Given a cycle $H$, we define a \textit{trivial chord} as one that joins vertices that are distance two apart on $H$.

\textbf{Lemma 9} Let $G$ be a hamiltonian outerplanar graph with outer cycle $H$ such that all chords of $H$ if any are trivial. Let $v$ be a designated vertex. One can color some of the vertices of $G$ blue or red such that both the red and the blue vertices form an independent dominating set of $G$. Further, one can specify whether $v$ is colored or not, except when $v$ is part of a restrictive pair when it has to be colored.

\textbf{Proof.} Assume first that $G$ has no chords. Then the claim is straight-forward if $v$ is required to be colored, as one can color all but at most one vertex. If $v$ is required to be uncolored and the cycle has odd length, then alternate colors on the cycle omitting $v$. Length 4 would mean $v$ is in a restrictive pair; so the length of an even cycle must be at least 6. Then one can properly color all vertices but $v$ and its antipodal vertex $v'$ such that both $v$ and $v'$ have neighbors of both colors.

The result is also immediate if $G$ is $K_4 - e$. So assume otherwise. Let $T$ denote the set of vertices of degree 2 that lie in a triangle. Then let $C$ be the subgraph $G - T$; it is a cycle consisting of the trivial chords of $H$ and segments of $H$.

Our basic strategy is to provide a partial proper red-blue coloring such that (a) every vertex in $C \cup \{v\}$ has both colors in its closed neighborhood, and (b) for every other vertex at least one neighbor is colored. Let us call such a coloring \textit{$C$-worthy}. One can complete a $C$-worthy coloring to the claim of the lemma as follows: repeatedly choose a vertex that is missing a color and give it the color it is missing.
Assume $v$ is required to be colored. If $v$ is on $C$, then color it and then alternate red-blue going round the cycle. If $C$ is even all vertices are colored; if $C$ is odd then one vertex is not colored but its two neighbors have different colors. In either case the coloring is $C$-worthy. If $v$ is in $T$, then properly color the cycle so that all but one vertex $x$ is colored, where vertex $x$ is a neighbor of $v$. Then give $v$ the color it is missing. Again the coloring is $C$-worthy.

Assume $v$ is required to be uncolored. If $v$ is in $T$, then color its two neighbors and as much of the rest of $C$ as can be properly colored. Again the coloring is $C$-worthy.

It remains to consider the case that $v$ is on $C$. If $v$ has a neighbor off $C$, then color one such neighbor and then properly color all vertices on $C$ except $v$. If $v$ has no neighbor off $C$, then it has degree 2 in $G$. Color one neighbor red and one blue, and then color as many vertices of $C$ as possible. If $C$ has odd length, then we get all but $v$ colored and have a $C$-worthy coloring. If $C$ has even length and length at least 6, we can ensure that the second uncolored vertex is far enough away from $v$ so that the coloring is $C$-worthy. If $C$ is a 4-cycle, then since $v$ is being required uncolored, it is not part of a restrictive pair. It follows that both of its neighbors have degree more than 2 and it is easily checked that the only possibility for $G$ has order 6 and there is a suitable coloring. QED

We next consider 2-connected outerplanar graphs.

**Lemma 10** Let $G$ be a 2-connected outerplanar graph and $v$ a designated vertex. One can color some of the vertices blue or red such that both the red and the blue vertices form an independent dominating set of $G$. Further, one can specify whether $v$ is colored or not, except when $v$ is part of a restrictive pair when it has to be colored.

**Proof.** The proof is by induction. Let $H$ be the outer cycle of $G$. The base case is that all chords of $H$ if any are trivial, which is covered by the above lemma. So we may assume there exists a nontrivial chord.

Now, consider all ordered pairs $(x, y)$ of vertices such that $x$ and $y$ are joined by a nontrivial chord $c$ and the segment of $S$ of $H$ from $x$ to $y$ does not contain $v$ as an
interior vertex. Out of all such pairs, choose \((x, y)\) so that \(S\) is as short as possible. It follows that there is no other nontrivial chord with both ends in \(S\).

Let \(G'\) be the \(\{x, y\}\)-component of \(G\) containing \(v\). That is, the subgraph induced by all vertices that can reach \(v\) without going through \(x\) or \(y\) first. If \(v\) is now in a restrictive pair when it wasn’t before, then that means that the original graph was \(C_6\) with one diametrical chord containing \(v\). It is easily checked that such a graph has a valid coloring. Otherwise we may apply induction to \(G'\) with designated vertex \(v\).

It remains to dominate the vertices of \(S\). We proceed as in the base case described in the above lemma. Let \(T\) be the vertices of degree 2 in triangles; color as many of the other vertices as possible respecting the coloring of the chord \(c\). This provides the equivalent of a \(C\)-worthy coloring. QED

Finally we are in a position to prove the main result about outerplanar graphs:

**Theorem 11** If \(G\) is an outerplanar graph with minimum degree at least 2, then \(G\) has two disjoint independent dominating sets.

**Proof.** The proof is by induction (but not with any specification hypothesis). If the graph is 2-connected, then the result follows from the above lemma. So assume there is a cut-vertex. Then there is an end-block \(B\) with a unique cut-vertex \(v\). Let \(G'\) be the graph \(G - (B - \{v\})\).

The first case is that \(v\) has degree at least two in \(G'\). Then we can apply the induction hypothesis to \(G'\) to produce a coloring with two disjoint independent dominating sets. Then using the above lemma, color \(B\) with \(v\) designated colored or not as it is in the coloring of \(G'\). The only problem that may arise is that \(v\) is in a restrictive pair in the subgraph \(B\), but \(v\) is not colored in \(G'\). Then consider \(B'\) to be the graph \(B\) with both \(v\) and (one of) its degree 2 neighbor \(w\) removed. Let \(x\) be \(w\)'s other neighbor. Apply induction to \(B'\) with \(x\) being prescribed in. Add \(w\) to the independent dominating that \(x\) is not in, and we have the desired coloring.

The second case is that \(v\) has degree one in \(G'\). Then consider the bridge out of \(v\) and extend it to a path \(P\) from \(v\) to \(w\) where \(w\) has degree more than 2 in \(G\) while every internal vertex if any has degree 2 in \(G\). Remove the internal vertices of the path \(P\) from \(G - B\) and induct on that. Apply the above lemma to \(B\) with
v designated as colored. Then color the vertices on the interior of P to finish. (If P has no interior vertex we can assume v and w have different colors, if w is colored.) This completes the proof.  

QED

5 Graph Families and Operations

We consider the behavior of the parameters for several graph operations and graph families.

5.1 Union and join

The results for the disjoint union and join are straight-forward:

**Lemma 12** For the disjoint union, \( \text{FIDOM}(G+H) = \min\{\text{FIDOM}(G), \text{FIDOM}(H)\} \).

For the join, \( \text{FIDOM}(G \oplus H) = \text{FIDOM}(G) + \text{FIDOM}(H) \).

A slightly more general result than the disjoint union is the following result. Let \( B(G, H, g, h) \) be a graph produced from disjoint copies of graph \( G \) and \( H \) by adding a bridge \( gh \) between vertex \( g \) of \( G \) and vertex \( h \) of \( H \).

**Lemma 13** Assume graphs \( G \) and \( H \) have \( \text{FIDOM}(G), \text{FIDOM}(H) \geq r \geq 2 \). Then it holds that \( \text{FIDOM}(B(G, H, g, h)) \geq r \).

**Proof.** For \( G \) there is a family \( F_G \) of independent dominating sets and similarly a family \( F_H \) for \( H \). By duplicating if needed, we may assume that \( |F_G| = |F_H| \).

We can then index \( F_G = \{F^i_G\} \) so that the early sets contain vertex \( g \), and index \( F_H = \{F^i_H\} \) so that the later sets contain vertex \( h \). Then we get family \( F \) by taking \( F^i_G \cup F^i_H \) for each \( i \). Note that this set never includes both \( g \) and \( h \), since each is in at most half the (sets of their) family. Thus each set \( F^i_G \cup F^i_H \) is independent. Furthermore, \( m(F) = \max(m(F_G), m(F_H)) \).  

QED
5.2 Lexicographic product

Lemma 14 For the lexicographic product of graphs $G$ and $H$, $\text{FIDOM}(G[H]) = \text{FIDOM}(G) \times \text{FIDOM}(H)$.

Proof. It is easy to see (and known) that an independent dominating set $J$ of $G[H]$ has a projection onto $G$ that is an independent dominating set of $G$ and for each copy of $H$ that contains a vertex of $J$, the set $J$ restricted to that copy is an independent dominating set.

So consider an optimal family $\mathcal{F}_G = \{ F^i_G : 1 \leq i \leq r \}$ of independent dominating sets of $G$ and an optimal family $\mathcal{F}_H = \{ F^i_H : 1 \leq i \leq s \}$ of independent dominating sets of $H$, and define $\mathcal{F} = \{ F^i_G \times F^j_H : 1 \leq i \leq r, 1 \leq j \leq s \}$. Each set is an independent dominating set of $G[H]$; further, a vertex is in a set of $\mathcal{F}$ at most $m(\mathcal{F}_G) \times m(\mathcal{F}_H)$ times; thus $\text{FIDOM}(G[H]) \geq rs/(m(\mathcal{F}_G) \times m(\mathcal{F}_H)) = \text{FIDOM}(G) \times \text{FIDOM}(H)$. On the other hand, if we take any family $\mathcal{F}'$ of minimal dominating sets of $G[H]$, then some copy of $H$ must be used at least $|G|/\text{FIDOM}(G)$ proportion of the time, and within that some vertex must be used at least $|H|/\text{FIDOM}(H)$ proportion of the time; thus $|\mathcal{F}'|/m(\mathcal{F}') \leq \text{FIDOM}(G) \times \text{FIDOM}(H)$. QED

5.3 Cubic graphs

Berge showed that every cubic graph has two disjoint independent dominating sets. (For a proof and some background, see [8]. Indeed in [8] we showed that every graph with minimum degree at least 2 and maximum degree at most 3 has two disjoint independent dominating sets and posed as a question whether that result extends to allowing maximum degree 4.)

There are many cubic graphs that have only two disjoint independent dominating sets. For example, any cubic graph $G$ with $i(G) > n/3$. Computer search suggest that maybe $\text{FIDOM}(G) \geq 5/2$ for every cubic graph $G$, except for $K_{3,3}$. If true, the value 5/2 would be best possible because of the prism $C_5 \Box K_2$ discussed earlier.

Even if one adds the condition of being bipartite and planar, there are examples of cubic graphs with only two disjoint independent dominating sets. The smallest such graph $J$ has 16 vertices and is shown below. Also, Abrishami et al. [2] conjectured that $i(G) \leq n/3$ for any bipartite cubic planar graph of order $n$. We remark
that $\text{FIDOM}(J) = 3$, and indeed that $\text{FIDOM}(G) \geq 3$ for all small bipartite cubic planar graphs.

![Figure 8. A planar bipartite cubic graph $J$ with $\text{idom}(J) = 2$](image)

What happens in general regular graph $G$ is unclear. Payan [15] showed that being regular does not imply $\text{idom}(G) \geq 2$; but maybe it is true that $\text{FIDOM}(G) \geq 2$ always.

6 Other Questions

Apart from the open problems discussed above, there is also the question of the complexity. It is known that testing whether a graph has two disjoint independent sets is NP-hard (see, for example, Theorem 7.1 in [9]). But it is unclear what the situation for the fractional independent domination number is.

References


