Eccentric Counts, Connectivity and Chordality

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Abstract

Let \( e_j \) denote the number of vertices of eccentricity \( j \) in a graph. We provide sharp lower bounds on \( e_j \) in graphs of given connectivity and in chordal graphs of given connectivity.

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1 Introduction

The eccentricity of a vertex \( v \) is the maximum distance of a vertex from \( v \), and the eccentricity sequence of a graph is the sequence of the eccentricities of its vertices (sorted in nondecreasing order). In 1975, Lesniak [7] characterized the sequences of integers that are the eccentric sequences of trees. However, a characterization of eccentric sequences for graphs in general has not been found. Indeed, partial results suggest that the problem is complex (see, for example, [5, 6]). We consider here a related question, namely, bounds on the number of vertices with a given eccentricity. In this regard, Mubayi and West [8] gave sharp lower bounds and almost sharp upper bounds on the number of vertices of given eccentricity in terms of order and diameter.

We will denote the eccentricity of vertex \( v \) by \( \text{ecc}(v) \). We use \( E_j \) for the set of vertices of eccentricity \( j \), and we let \( e_j = |E_j| \) (the \( j^{th} \) eccentricity count). Further, the radius \( r \) is the minimum eccentricity and the diameter \( d \) is the maximum eccentricity.

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It is easy to show (see for example [7]) that in a connected graph, \( e_j \geq 2 \) for all \( j \) with \( r < j \leq d \). Indeed, since \( E_j \) is a cut-set when \( r < j < d \), it follows that

For \( r < j < d \), \( e_j \geq k \) in a \( k \)-connected graph.

In this paper we consider improvements on this result for some graph classes. In section 2 we provide a sharp lower bound on \( e_j \) in \( k \)-connected graphs, and in section 3 we provide a sharp lower bound on \( e_j \) in \( k \)-connected chordal graphs. Finally, in section 4 we consider the question for a generalization of chordal graphs.

## 2 Connectivity

We begin with a lower bound for the eccentricity count \( e_j \) in general graphs based on the connectivity. The results shows that in a \( k \)-connected graph, the eccentricity count \( e_j \) can be less than \( 2k \) only if \( j \) is not too close to \( 2r \). As a corollary we obtain a lower bound on \( e_j \) in terms of connectivity.

**Theorem 1.** Let \( G \) be a \( k \)-connected graph with radius \( r \) and diameter \( d \), and let \( j \) be such that \( r < j < d \). If \( e_j = s \) and \( s < 2k \), then

\[
    j \leq \frac{s}{s + 1} 2r + \frac{s - 1}{2(s + 1)},
\]

and this is sharp.

**Proof.** Let \( e_j = s \) and let \( E_j = \{a_1, \ldots, a_s\} \). Let \( F \) be the set of vertices of \( G \) with eccentricity more than \( j \). Since \( j < d \), this set is nonempty. Every vertex of \( F \) is within distance \( r \) of the center, while every vertex of \( E_j \) is at least distance \( j - r \) from the center. Thus

"every vertex of \( F \) is within distance \( r - (j - r) = 2r - j \) of \( E_j \)."

For \( 1 \leq i \leq s \), define \( F_i \) as the set of vertices in \( F \) that are within distance \( 2r - j \) of \( a_i \) via a path that does not contain any other vertex of \( E_j \). Note that \( \bigcup_i F_i = F \).
Now, define an auxiliary graph $H$ as follows. Create one vertex $w_i$ for each set $F_i$ that is nonempty. Then add an edge from $w_i$ to $w_{i'}$, if there is a vertex in $F_i \cap F_{i'}$ or there is an edge from $F_i$ to $F_{i'}$.

We claim that $H$ is connected. For, let $H'$ be any component of $H$. Note that by the way we defined $F_i$, each induced subgraph $\langle F_i \cup \{a_i\} \rangle$ of $G$ is connected. Further, if $x \in F_i$ is adjacent to some $a_i$, then $x \in F_{i'}$, so that $w_i$ is a vertex in $H$ and there is an edge in $H$ from $w_i$ to $w_{i'}$. It follows that $\{ a_i : w_i \in H' \}$ is a cut-set in $G$. By the connectivity of $G$, it follows that this set has at least $k$ vertices, and so $H'$ has at least $k$ vertices. Since $H$ has less than $2k$ vertices, it follows that $H$ is connected.

Now, let $R$ be the radius of $H$ and let $w_{i^*}$ be a central vertex in $H$. Let $x$ be any vertex of $F_{i^*}$ adjacent to $a_{i^*}$. We can bound the eccentricity in $G$ of $x$ as follows.

Let $y$ be an eccentric vertex for $x$ (that is, $y$ is farthest from $x$). We know that $y \in F_i$, since all other vertices have eccentricity at most $j$. Say $y \in F_{i'}$. Then a path in $G$ from $x$ to $y$ can be constructed within the induced subgraph $\langle F \cup E_y \rangle$ by mimicking the path in $H$ from $w_{i^*}$ to $w_{i'}$. Since each induced subgraph $\langle F_i \cup \{a_i\} \rangle$ has diameter at most $2(2r - j)$, vertex $x$ has eccentricity at most $1 + (2r - j)$ within the induced subgraph $\langle F_{i^*} \cup \{a_{i^*}\} \rangle$, and $H$ has radius $R$, it follows that

$$
\text{ecc}(x) \leq 1 + (2r - j) + R(2(2r - j) + 1).
$$

But, $\text{ecc}(x) \geq j + 1$, because $x \in F_i$. It follows that

$$
j \leq \frac{r(4R + 2) + R}{2R + 2}.
$$

Note that this upper bound is increasing in $R$. If we set $R = (s - 1)/2$ into the upper bound and simplify, then we obtain the formula in the statement of the theorem. Thus, if $R \leq (s - 1)/2$, we are done.

So assume that $R > (s - 1)/2$. This means that $R = s/2$. It follows that $s$ is even and $H$ is a path on $s$ vertices. Therefore, there are two central vertices in $H$, say $w_{i^*}$ and $w_{i^{**}}$. Since these two vertices are adjacent in $H$, there is a vertex $x'$ in $F_{i^*}$ that is either in $F_{i^{**}}$ or adjacent to a vertex of $F_{i^{**}}$. 

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By a similar argument to the above, it follows that
\[ \text{ecc}(x') \leq \frac{(s/2)(2(2r - j) + 1)}{s + 1}. \]
And since \( \text{ecc}(x') \geq j + 1 \), we get that
\[ j \leq \frac{2sr + (s/2 - 1)}{s + 1}, \]
as required. (The bound for \( R = (s - 1)/2 \) appears larger, but as \( s \) is even, that bound is not an integer and is not exactly attainable.) This proves the theorem.

**Corollary 2.** Let \( G \) be a \( k \)-connected graph with radius \( r \) and diameter \( d \), and let \( j \) be such that \( r < j < d \). Then
\[ e_j \geq \left[ \frac{2j + 1}{4r - 2j + 1} \right]^{2k}, \]
where by \([A]^{2k}_k\) we mean \( \min\{\max\{A, k\}, 2k\} \).

Theorem 1 is best possible. First we note that it is easy to construct \( k \)-connected graphs where \( e_j = 2k \). For example, take the power \( P_n^k \) of the path for \( n \) sufficiently large.

Further, the upper bound on \( j \) is best possible in general. Here is an example of an extremal graph. Fix positive integers \( r, j, s, k \) such that \( s \geq 2 \), \( r < j < 2r - 1 \), and \( j \leq (2sr + (s - 1)/2)/(s + 1) \). Construct graph \( G(r, j, s, k) \) as follows.

Start with a rooted tree with root \( c \) where \( c \) has \( s \) neighbors, \( a_1, \ldots, a_s \), each of which has two leaf-neighbors \( b_i \) and \( b'_i \). Then subdivide each edge \( ca_i \) to have length \( j - r \) and subdivide every other edge to have length \( 2r - j \). Add an edge from \( b'_i \) to \( b_{i+1} \) for each \( 1 \leq i \leq s - 1 \). Add a path of length 3 joining the two child neighbors of \( a_i \) for each \( 1 \leq i \leq s \). Finally, expand every vertex \( v \) except the \( a_i \) into a clique \( C_v \) of order \( k \) in the natural way: a vertex in \( C_v \) is adjacent to a vertex in \( C_{v'} \) if and only if \( v \) and \( v' \) were adjacent, and a vertex in \( C_v \) is adjacent to \( a_i \) if and only if \( v \) and \( a_i \) were adjacent.

Note that vertices in \( C_c \) have eccentricity \( r \), and so \( a_i \) has eccentricity at most \( j \). Let \( T = \{b_1, b'_1\} \), and for any vertex \( v \), define \( X(v) = \max\{d(v, b_1), d(v, b'_1)\} \). Consider any of the \( a_i \). It follows that the shortest path in \( G - c \) from \( a_i \) to the
farther vertex of $T$ has length at least $s(2r-j) + (s-1)/2$. By the upper bound on $j$, it follows that $X(a_i) \geq j$, and so $\text{ecc}(a_i) = j$.

Every vertex on the subdivided edge $ca_i$ has eccentricity less than $j$, since it is at distance less than $j-r$ from $C_c$. Further, let $w$ be any descendant of the $\{a_i\}$. Any path from $w$ to a vertex of $T$ that goes via $C_c$ has length more than $j$. Further, $X(w) > X(a_i)$ for some $a_i$, since the shortest $C_c$-avoiding path from $w$ to the farther vertex of $T$ meets the shortest $C_c$-avoiding path from some $a_i$ to that vertex of $T$ at a point closer to $a_i$ than to $w$. It follows that $\text{ecc}(w) > j$.

Hence $E_j = \{a_1, \ldots, a_s\}$. At the same time, $G(r,j,s,k)$ has connectivity $\min\{s,k\}$. In Figure 1 we depict $G(10,17,5,k)$.

![Figure 1: The graph $G(10,17,5,k)$ with $e_{(5r+1)/3} = 5$.](image)

In passing, we note that there is no equivalent result for edge connectivity. That is, one can readily construct graphs with arbitrary edge connectivity and still have $e_j = 2$ for specific $j$. For example, consider the graph obtained from a path of suitable length by expanding every vertex except the two vertices of eccentricity $j$ into a large clique.
3 Chordal Graphs

Recall that a graph is chordal if every cycle of length at least 4 has a chord. We will need the following well-known facts about chordal graphs (a separator separates some pair of nonadjacent vertices) (see, for example, [1]):

**Proposition 3.** If $G$ is chordal graph, then
(a) $G$ has a simplicial vertex;
(b) every minimal separator is a clique.

We will also need the following simple result, which is probably known.

**Lemma 4.** Let $G$ be a $k$-connected chordal graph, and $S$ a cut-set of $G$. Then in each component of $G - S$ there is a vertex $v$ such that $N(v) \cap S$ contains a $k$-clique.

**Proof.** Recall the well-known fact that for any graph $G$, if vertex $w$ is simplicial and $G$ is not complete, then the removal of $w$ cannot decrease the connectivity.

Consider any simplicial vertex $w$. If $w \in S$, then $w$ has neighbors in at most one component of $G - S$. So we can induct on $G - w$; the vertex $v$ so found has the desired property in $G$ also.

So assume $w \notin S$. If $w$ is not an isolated vertex in $G - S$, then again we can induct on $G - w$. But if $w$ is an isolated vertex in $G - S$, then its at least $k$ neighbors are in $S$ and form a clique. □

**Theorem 5.** Let $G$ be a $k$-connected chordal graph with radius $r$ and diameter $d$, and let $j$ be such that $r < j < d$. Then

$$e_j \geq 2k,$$

and this is sharp.

**Proof.** Say $E_j = \{a_1, \ldots, a_s\}$ and suppose $s < 2k$. Let $F$ be the set of vertices of $G$ with eccentricity more than $j$. By the above lemma, there exists a vertex $x$ in $F$ that has $k$ neighbors in $E_j$. 

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Let $c$ be a central vertex and let $y$ be an eccentric vertex for $x$. Then $y \in F$. Since the set $E_j$ separates vertex $y$ from $c$, there is a minimal subset $C_y$ of $E_j$ separating $y$ from $c$. By chordality, $C_y$ is a clique. Further, $y$ is within distance $r$ of $c$, while every vertex of $E_j$ is at least distance $j-r$ from $c$, and so $y$ is within distance $r-(j-r) = 2r-j$ of $C_y$.

Now, since $s < 2k$, it follows that $N(x)$ intersects the clique $C_y$. Thus $d(x,y) \leq 2 + (2r-j)$, and so $\text{ecc}(x) \leq 2 + 2r - j$. But $\text{ecc}(x) \geq j + 1$. It follows that $j \leq r + 1/2$, a contradiction.

It is easy to construct $k$-connected chordal graphs where $e_j = 2k$. For example, take the power $P_n^k$ of the path for $n$ sufficiently large, as before.

### 4 Chordal Generalizations

The key ingredient in the proof of Theorem 5 is that a minimal separator is a clique. One can generalize this theorem to graphs where there is some upper bound on the diameter of a minimal separator (diameter measured in the original graph). One such family is the $k$-chordal graphs. For $k \geq 2$, a graph is $k$-chordal if there is no induced cycle with length bigger than $k$ (so that chordal graphs are 3-chordal and trees are 2-chordal).

Chepoi et al. [4] observed that:

**Proposition 6.** If graph $G$ is $\ell$-chordal, then the diameter of any minimal separator is at most $\ell/2$.

Using this, we can establish the following result:

**Theorem 7.** If $G$ is a $k$-connected $\ell$-chordal graph, then

(a) $d \geq 2r - \ell - 1$.
(b) $e_j \geq 2k$ for $d > j > r + (\ell + 1)/2$.

**Proof.** The result is immediate if $G$ is complete; so assume otherwise. Let $x$ and $y$ be diametrical vertices. Let $X$ be the vertices within distance $\lceil d/2 \rceil - 1$ of $x$, and $Y$ the vertices within distance $\lceil d/2 \rceil - 1$ of $y$. Then $V - (X \cup Y)$ is a separator, and it contains a minimal $(x,y)$-separator $S$. 

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Every vertex $v$ in the graph is within distance $d$ of both $x$ and $y$. Since the shortest path from $v$ to at least one of $x$ or $y$ must go through $S$, it follows that vertex $v$ is within distance $d - \lfloor d/2 \rfloor$ of $S$. Thus every vertex in $S$ has eccentricity at most $[d/2] + \ell/2$. This yields part (a).

Further, consider the $|S| \geq k$ internally disjoint paths between $x$ and $y$. These paths go through $S$, where their vertices have eccentricity at most $[d/2] + \ell/2$. So, assume $j > r + (\ell + 1)/2$. It follows that each such path must contain two vertices of eccentricity $j$, so that $e_j \geq 2k$. \qed

The above bound is probably not sharp in terms of $\ell$, since for example, it is known that $d \geq 2r - 2$ in chordal graphs (see [3, 2, 9]). Nevertheless, in part (b) some lower bound on $j$ is necessary to force $e_j \geq 2k$, as the results in section 2 showed.

5 Open Questions

The ultimate goal is to characterize eccentric sequences of graphs. In order to make progress, perhaps one can obtain improved bounds for $e_j$ for other graph classes, or even characterize the eccentric sequences of some graph classes.

References


