The Disjunctive Domination Number of a Graph

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Abstract

For a positive integer \( b \), we define a set \( S \) of vertices in a graph \( G \) as a \( b \)-disjunctive dominating set if every vertex not in \( S \) is adjacent to a vertex of \( S \) or has at least \( b \) vertices in \( S \) at distance 2 from it. The \( b \)-disjunctive domination number is the minimum cardinality of such a set. This concept is motivated by the concepts of distance domination and exponential domination. In this paper, we start with some simple results, then establish bounds on the parameter especially for regular graphs and claw-free graphs. We also show that determining the parameter is NP-complete, and provide a linear-time algorithm for trees.

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1 Introduction

Among the many variants of domination, a few consider the distance that a vertex is from the set. For example, in distance domination, a vertex "dominates" all those vertices within a specific distance of it. Recently, Dankelmann et al. \cite{2} considered the case where the domination of a vertex reduces as distance increases. Motivated by these ideas, we consider here the situation of a set $S$ where each vertex is either dominated by $S$ or has sufficiently many members of $S$ close by.

We start with some known parameters. A set $S$ dominates vertex $v$ if $v$ is either in $S$ or adjacent to (joined by an edge to) some vertex of $S$. For a graph $G$, a set $S \subseteq V(G)$ is a dominating set of $G$ if every vertex not in $S$ is adjacent to $S$. The domination number, $\gamma(G)$, is the minimum cardinality of a dominating set.

Dankelmann et al. \cite{2} recently defined exponential domination. Let $G$ be a graph and $S \subseteq V(G)$. For each vertex $u \in S$ and for each $v \in V(G) - S$, we define $\bar{d}(u,v)$ to be the length of a shortest $u$–$v$ path in $G - (S - \{u\})$ if such a path exists, and $\infty$ otherwise. If, for each $v \in V(G) - S$ we have $\sum_{u \in S} \frac{1}{2^{\bar{d}(u,v) - 1}} \geq 1$, then $S$ is an exponential dominating set. The smallest cardinality of an exponential dominating set is the exponential domination number, $\gamma_{\text{exp}}(G)$. One can think of this in the following way: each vertex dominates its neighbors, $\frac{1}{2}$-dominates those at distance 2, and so on. The parameter $\gamma_{\text{exp}}$ seems difficult to calculate. Some further results on it were given in \cite{1}.

A different direction was recently taken by Hedetniemi et al. \cite{6} who require a vertex to be dominated and to have multiple vertices within distance two. In this paper we consider a parameter motivated by all of the above.

**Definition 1** For a graph $G$ and positive integer $b$, define a set $S \subseteq V(G)$ to be a $b$-disjunctive dominating set ($b$DDS) of $G$, if every vertex $v$ not in $S$ is either adjacent to $S$ or there are at least $b$ vertices of $S$ within distance two of $v$ (or both). The $b$-disjunctive domination number, $\gamma^d_b(G)$, is the minimum cardinality of a $b$-disjunctive dominating set.

For example, consider the Petersen graph. This graph has a 2-disjunctive domination number of 2, as realized by any pair of vertices. For $b \geq 3$, the
$b$-disjunctive domination number is 3, as realized by any minimum dominating set.

Now, the parameter $\gamma^d_1$ is the distance domination number. This parameter has been studied for example in [7]; so for the rest of the paper we will look solely at the situation where $b \geq 2$.

We start with some simple results, then establish more bounds, especially for regular graphs and claw-free graphs. We conclude by showing that determining the parameter is NP-complete, and provide a linear-time algorithm for trees.

## 2 Some Simple Results

It is immediate that:

**Lemma 1** For any graph $G$ and $b \geq 1$,

(a) $\gamma^d_b(G) \leq \gamma^d_{b+1}(G)$;
(b) $\gamma^d_b(G) \leq \gamma(G)$;
(c) If $b \geq 2$ then $\gamma^d_b(G) \geq \gamma_{\text{exp}}(G)$.

**Proof:** Any set that dominates $G$ is a bDDS. So, $\gamma^d_b(G) \leq \gamma(G)$. For $b \geq 2$, any bDDS is also an exponential dominating set. So, $\gamma_{\text{exp}}(G) \leq \gamma^d_b(G)$. □

We next note the values for extreme cases. It is easy to show that in a graph $G$ on $n$ vertices, $\gamma^d_b(G) = n$ if and only if $G$ is empty. Further, for $b \geq 2$, $\gamma^d_b(G) = 1$ if and only if $\gamma(G) = 1$. From this, one obtains $\gamma^d_b$ for the complete bipartite graph:

**Lemma 2** For all $b \geq 2$, $n, m \geq 2$, $\gamma^d_b(K_{n,m}) = \gamma(K_{n,m}) = 2$.

We consider further the connection between our parameter and the ordinary domination number.

**Lemma 3** For any graph $G$ and $b \geq 2$,

(a) If $\gamma(G) \leq b$, then $\gamma^d_b(G) = \gamma(G)$;
(b) If $G$ has diameter at most 2, then $\gamma^d_b(G) \leq b$;
(c) If $G$ has maximum degree $\Delta(G)$ and $\Delta(G)(\Delta(G) - 1) < b$, then $\gamma^d_b(G) = \gamma(G)$.  

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Proof: (a) Any set $S$ that is a bDDS but not dominating must have at least $b$ vertices.

(b) In this case, any set of $b$ vertices is a bDDS.

(c) Assume $\Delta(G)(\Delta(G) - 1) < b$. Then, for any vertex $v \in V(G)$, there are fewer than $b$ vertices at distance exactly two from $v$. So, every bDDS must be a dominating set. □

We observe that the bound on the maximum degree is sharp, in that one can find graphs with $b = \Delta(G)(\Delta(G) - 1)$ and $\gamma^d_b(G) < \gamma(G)$. For example, $\Delta(C_8)(\Delta(C_8) - 1) = 2$, and $\gamma^d_2(C_8) = 2 < 3 = \gamma(C_8)$.

We finish this section with a monotonicity result. It is easy to observe that the deletion of an edge can increase the domination number by at most 1. Surprisingly, Dankelmann et al. [2] showed that this is not the case with exponential domination. We show that the analogous result does hold for b-disjunctive domination.

**Theorem 4** For any graph $G$ and $b \geq 2$, $\gamma^d_b(G - e)\leq \gamma^d_b(G) + 1$ for any edge $e$.

Proof: Let $e = uv$ and let $S$ be a bDDS of $G$. There are three cases.

Case 1: Both $u$ and $v$ are in $S$. Then $S$ is still a bDDS of $G - e$.

Case 2: Exactly one of $u$ and $v$ is in $S$, say $u$. Then $S \cup \{v\}$ is a bDDS of $G - e$.

Case 3: Neither $u$ nor $v$ is in $S$. Then the only vertices that might not be sufficiently dominated by $S$ in $G - e$ are $u$ and $v$. So, assume $u$ is insufficiently dominated. Then in $G$, it must be the case that there is a vertex $w \in S$ at distance 2 from $u$ such that $e$ is on the shortest path from $u$ to $w$. It follows that $w$ is a neighbor of $v$, and so $S \cup \{u\}$ is a bDDS of $G - e$. □

3 Bounds and Families

3.1 Cycles and paths

We next determine $\gamma^d_b$ for cycles.
**Theorem 5** (a) For $b = 2$,

$$\gamma_d^b(C_n) = \gamma_{\exp}(C_n) = \begin{cases} 2 & n = 4 \\ \lceil n/4 \rceil & n \neq 4. \end{cases}$$

(b) For $b \geq 3$,

$$\gamma_d^b(C_n) = \gamma(C_n) = \lceil n/3 \rceil.$$  

**Proof:** (a) Dankelmann et al. [2] calculated that $\gamma_{\exp}(C_n) = 2$ if $n = 4$ and $\lceil n/4 \rceil$ otherwise. So by Lemma 1, it suffices to show that this value is an upper bound for 2-disjunctive domination. The case $n = 4$ follows from Lemma 2, since $K_{2,2} = C_4$. Otherwise, construct $S$ by starting with some vertex $v$ and taking every 4th vertex. Then $|S| = \lceil n/4 \rceil$, and every vertex not in the closed neighborhood $N[S]$ has two vertices of $S$ at distance two from it.

(b) This follows from Lemma 3. □

The following theorem can be proved by a similar argument.

**Theorem 6** (a) For $b = 2$, $\gamma_d^b(P_n) = \gamma_{\exp}(P_n) = \lceil (n + 1)/4 \rceil$ for all $n$.

(b) For $b \geq 3$, $\gamma_d^b(P_n) = \gamma(P_n) = \lceil n/3 \rceil$ for all $n$.

### 3.2 Lower bounds from maximum degree

We give a general lower bound on the $b$-disjunctive domination number based on the maximum degree:

**Theorem 7** For any $b \geq 1$ and graph $G$ of order $n$, if each vertex has at most $\Delta$ neighbors and at most $D$ vertices at distance 2, then

$$\gamma_d^b(G) \geq \frac{n}{\frac{1}{b} + \Delta + D/b}.$$  

**Proof:** Let $S$ be a $b$DDS. A vertex $v \in S$ dominates itself and its neighbors, but can contribute only $1/b$ to the “domination” of each vertex at distance 2 from it. There are at most $D$ such vertices. □
Corollary 8  For any $b \geq 1$ and graph $G$ of order $n$ and maximum degree $\Delta$,  
\[ \gamma_b^d(G) \geq \frac{n}{1 + \Delta + \Delta(\Delta - 1)/b}. \]

It is possible to construct infinitely many graphs that achieve this lower bound. For example, let $r \geq b \geq 2$ and $a \geq 1$, and we define a graph $G_b(r,a)$ as follows. We start with $ab$ disjoint copies of the star $K_{1,r}$, and any $(r - b)$-regular graph $H$ on $ar(r - 1)$ vertices. Let $X$ denote the set of leaves of the stars and $Y = V(H)$. Then add edges such that every vertex in $X$ has $r - 1$ neighbors in $Y$, and every vertex in $Y$ has $b$ vertices in $X$ and these neighbors are in different stars. For example, one possibility for $G_2(3,1)$ is illustrated in Figure 1. It follows that the resultant graph $G_b(r,a)$ is $r$-regular, and the original $ab$ star-centers form a (minimum) $b$DDS of $G_b(r,a)$.

3.3 Graphs with maximum $\gamma_b^d$

Given a graph $G$, the corona $\text{Cor}(G)$ is formed by adding for each vertex $v \in V(G)$, a vertex $v'$, and joining it to $v$. It turns out that the $b$-disjunctive domination of the corona is given by the $b$-domination number of the original graph. For a graph $G$, a set $S \subseteq V(G)$ is a $b$-dominating set of $G$ if every vertex $v$ not in $S$ has at least $b$ neighbors in $S$. The $b$-domination number, $\gamma_b(G)$, is the minimum cardinality of a $b$-dominating set.

Theorem 9  For any graph $G$, $\gamma_b^d(\text{Cor}(G)) = \gamma_b(G)$.

Proof:  Consider a $b$DDS $S$ of $\text{Cor}(G)$, and let $L$ denote the leaves added in the formation of $\text{Cor}(G)$. If $v \in L \cap S$, then if we replace $v$ by its neighbor $w$ in $S$, we still have a $b$DDS of $\text{Cor}(G)$. So we may assume $S \subseteq V(G)$. 
Now consider any \( v \in L \). If its neighbor \( w \) is not in \( S \), then there are at least \( b \) vertices of \( S \) within distance 2 of \( v \). In other words, at least \( b \) neighbors of \( w \) are in \( S \). That is, \( S \) is a \( b \)-dominating set of \( G \).

Conversely, any \( b \)-dominating set of \( G \) is a \( b \)DDS of Cor(\( G \)). The theorem follows. \( \square \)

The above result enables us to determine the maximum value of the \( b \)-disjunctive domination number.

**Theorem 10** Let \( b \geq 2 \), and let \( G \) be a connected graph with \( n \geq 2 \) vertices. Then \( \gamma^d_b(G) \leq n/2 \) with equality if and only if \( G \) is \( C_4 \) or Cor(\( H \)) where \( \Delta(H) < b \).

**Proof:** It is well-known that \( \gamma(G) \leq n/2 \), so by Lemma 1 the upper bound follows. Further, equality requires that \( \gamma(G) = n/2 \). By the results of [3, 10], this means that \( G \) is \( C_4 \) or a corona. By Theorem 9, \( \gamma^d_b(\text{Cor}(H)) = \gamma_b(\text{Cor}(H)) \). The results follows from the observation that \( \gamma_b(H) \) equals the order of \( H \) if and only if \( \Delta(H) < b \). \( \square \)

There are infinitely large examples of equality for \( b \geq 3 \); take, for example, the corona of any cycle. However, for \( b = 2 \), the only examples of equality in the above theorem are \( C_4 \), \( P_4 \), and \( K_2 \).

**Corollary 11** If \( G \) is a connected graph with \( n \geq 5 \), then \( \gamma^d_2(G) \leq (n - 1)/2 \).

This bound is sharp. Indeed, we can determine the graphs which achieve this bound.

In order to characterize the connected graphs \( G \) of order \( n \) with \( \gamma^d_2(G) = (n - 1)/2 \), we define the following three families of graphs. For \( t \geq 2 \), let \( T_t \) be the tree obtained from a star \( K_{1,t} \) by subdividing every edge once and let \( T \) be the family of all such trees \( T_t \). For \( t \geq 2 \), let \( G_t \) be the graph obtained from \( T_t \) by adding an edge joining two support vertices of \( T_t \), and let \( \mathcal{G} \) be the family of all such graphs \( G_t \). (A support vertex is one adjacent to a leaf.) Finally for \( t \geq 2 \), let \( T_t' \) be the tree obtained from \( T_t \) by deleting an edge \( uv \) incident
with the central vertex $v$ of $T_t$ and adding the edge $uw$ for some neighbor $w$ of $v$ different from $u$, and let $T'$ be the family of all such trees $T'_t$. The graph $G_4$ and the trees $T_4$ and $T'_4$ are illustrated in Figure 2. It is easily checked that $\gamma_2^d(G_t) = \gamma_2^d(T_t) = \gamma_2^d(T'_t) = t$.

![Figure 2: The graph $G_4$ and the trees $T_4$ and $T'_4$.](image)

We will also need the characterization of graphs with $\gamma(G) = (n - 1)/2$, determined by Randerath and Volkmann [11] and Xu et al. [13]. They showed that all sufficiently large graphs $G$ with $\gamma(G) = (n - 1)/2$ are constructed by taking an arbitrary corona and adding a small subgraph and some edges. The following theorem can be read out of their result:

**Theorem 12** Let $G$ be a graph of order $n \geq 9$ with $\gamma(G) = (n - 1)/2$. Then one can partition $V(G)$ into three sets $A$, $B$, $C$, such that $|A| \in \{1, 3, 5\}$, the subgraph $\langle A \rangle$ induced by $A$ is connected, $|B| = |C|$, and every vertex in $C$ is a leaf with a different neighbor in $B$.

**Theorem 13** Let $G$ be a connected graph of order $n \geq 7$. Then, $\gamma_2^d(G) = (n - 1)/2$ if and only if $G \in G \cup T \cup T'$.

**Proof:** Assume that $\gamma_2^d(G) = (n - 1)/2$. It follows that $\gamma(G) = (n - 1)/2$. Assume first that $n = 7$. Then by computer search, or by examining the characterization given in [13] more closely, it follows that there are three graphs with $\gamma_2^d(G) = 3$, namely $G_3$, $T_3$, and $T'_3$.

So assume $n \geq 9$. Then $G$ has the form given in Theorem 12.

If $|A| = 5$, then by the results of [13] there are five possibilities for $\langle A \rangle$. By considering each of these possibilities and the restrictions they prove on edges between $A$ and $B$, it follows that in each case one can extend $B$ to a 2DDS of $G$ by adding one vertex of $A$, and thus $\gamma_2^d(G) \leq (n - 3)/2$. 

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If $|A| = 3$, then by the results of [13], if $\langle A \rangle$ induces a $P_3$ then both ends of the $P_3$ have neighbors in $B$. It follows that $B$ is a 2DDS of $G$ unless only one vertex $v$ in $B$ has neighbor(s) in $A$. Since we are assuming $n \geq 9$ and the graph is connected, it follows that $v$ has a neighbor $w$ in $B$; and it can be shown that $B - \{v\}$ can be extended to a 2DDS of $G$ of cardinality $(n - 3)/2$ by adding one vertex of $A$.

So it must be that $|A| = 1$; say $A = \{x\}$. Now, if $H = \langle B \rangle$ has a vertex $v$ of degree two or more, then $B - \{v\}$ is a 2DDS of $G$ of cardinality $(n - 3)/2$. So $H$ has maximum degree at most 1. If $H$ has two nontrivial components, say containing neighbors $v_1$ and $v_2$ of $x$, then $B \cup \{x\} - \{v_1, v_2\}$ is a 2DDS of $G$ of cardinality $(n - 3)/2$. It follows that $H$ has at most one nontrivial component. Thus, $G$ is one of the three graphs given in the theorem. □

3.4 Grids

We consider grids next.

**Theorem 14** Let $G_{2,m}$ be the $2 \times m$ grid given by $P_2 \square P_m$. For all $m \geq 1$,

(a) $\gamma^d_2(G_{2,m}) = \lceil (m + 2)/3 \rceil$;

(b) $\gamma^d_b(G_{2,m}) = \lceil (m + 1)/2 \rceil$ for $b \geq 3$.

PROOF: (a) By Theorem 7, it follows that $\gamma^d_2(G_{2,m}) \geq (2m)/6 = m/3$. The actual lower bound follows from this and examination of the problem of dominating the corners (the vertices of degree 2). We omit the details. For a 2DDS, take one vertex in every third column, alternating rows, as well as one vertex in the last column if none is already taken there. See Figure 3 for the pattern.

(b) The stated value is the value of the domination number [8]. So it suffices to show that one cannot do any better than using a minimum dominating set. We omit the details. □

**Theorem 15** Let $G_{3,m}$ be the $3 \times m$ grid given by $P_3 \square P_m$. For all $m \geq 1$,

(a) $\gamma^d_2(G_{3,m}) = \lceil (m + 1)/2 \rceil$;

(b) $\gamma^d_3(G_{3,m}) = \lceil 5(m + 1)/8 \rceil$;

(c) $\gamma^d_b(G_{3,m}) = \lceil (3m + 1)/4 \rceil$ for $b \geq 4$. 
Proof: These results were proved by the software described in [5] and we do not have a human proof of the lower bounds. The pattern for the 2DDS is in Figure 4, and the general pattern for the 3DDS in Figure 5. For \( b \geq 4 \), the \( b \)-disjunctive domination number equals the domination number [8].

4 Claw-Free Graphs

Our aim in this section is to investigate whether the upper bound for \( \gamma_d^b \) of half the order can be improved if a connected claw-free graph is sufficiently large.

We will need the following family of graphs. For \( b \geq 2 \), let \( \text{ecor}(K_b) \) denote the graph obtained from \( \text{Cor}(K_b) \) by adding one “extension” vertex adjacent to
all of \( V(K_b) \). Then for \( t \geq 1 \) let \( G_{b,t} \) be the claw-free graph obtained from taking \( t \) copies of \( ecor(K_b) \) and adding edges such that the extension vertices form a clique. The graph \( G_{3,4} \) is illustrated in Figure 6.

\[
G_{3,4} \quad \text{Figure 6: The graph } G_{3,4}.\]

**Proposition 16** For \( b \geq 2 \) and \( t \geq 1 \), \( \gamma^d_b(G_{b,t}) = tb \).

**Proof:** Let \( G = G_{b,t} \) and let \( S \) be a bDDS of \( G \). Let \( F \) be any copy of \( ecor(K_b) \). Suppose \( |S \cap V(F)| < b \). Then some leaf \( v \) in \( F \) is not in \( N[S] \). It follows that there are \( b \) vertices of \( S \) within distance 2 of \( v \), and all of these must be in \( F \), a contradiction. Hence, \( |S \cap V(F)| \geq b \), and so \( \gamma^d_b(G) \geq tb \).

Conversely, the set of support vertices in \( G \) form a \( \gamma(G) \)-set, and so \( \gamma^d_b(G) \leq \gamma(G) = tb \). \( \square \)

### 4.1 Result for \( b = 2 \)

We are now in a position to prove the following result. Let \( G_3 \) denote the graph on 7 vertices mentioned in Theorem 13.

**Theorem 17** If \( G \) is a connected claw-free graph of order \( n \), then \( \gamma^d_2(G) \leq 2n/5 \), unless \( G \in \{K_1, P_2, P_4, C_4, G_3\} \).

**Proof:** The proof is by induction on \( n \). The result is automatic for \( n = 1 \) and \( n = 2 \), so assume \( n \geq 3 \). If \( \delta(G) \geq 2 \), then by the results of McCuaig and Shepherd [9], \( \gamma^d_2(G) \leq \gamma(G) \leq 2n/5 \) unless \( G \) is one of seven exceptional graphs. Only two of these exceptional graphs are claw-free: \( C_4 \) and \( C_7 \). The first is listed as an exception above; the second has \( \gamma^d_2(C_7) = 2 \). So we may assume that \( G \) has a leaf.
Let $u$ be a leaf and $v$ its neighbor. Let $X = N[v] \setminus \{u\}$. By the claw-freeness of $G$, the set $X$ is a clique. We call a component of $G - X$ a fragment, and for each fragment $F$ we choose a vertex $v_F \in X$ that is adjacent to a vertex of $F$. By the claw-freeness of $G$, every vertex in $X$ is adjacent to vertices from at most one fragment. In particular, the chosen vertices, $v_F$, associated with each fragment are distinct.

Let $X_1$ be those vertices in $X$ that are not one of the chosen vertices $v_F$ where $F \in \{P_2, P_4, C_4, G_3\}$. Thus if $x \in X_1$, then either $x$ is not a chosen vertex associated with a fragment or $x = v_F$ for some fragment $F$ where $F \notin \{P_2, P_4, C_4, G_3\}$. Let $Y$ be the set consisting of the vertices of the $K_1$-fragments together with the set $X_1$.

We now construct a set $S$ of vertices in $G$ as follows. Start with a minimum 2DDS, $T$ say, of the subgraph of $G$ induced by $Y$. Then for every fragment $F \neq K_1$ proceed as follows.

- If $F \notin \{P_2, P_4, C_4, G_3\}$, then choose a minimum 2DDS of $F$ (which by the inductive hypothesis has cardinality at most $2|V(F)|/5$).
- If $F = P_2$, then choose $v_F$ if it is adjacent to both vertices in $F$, and choose the vertex of $F$ adjacent to $v_F$ otherwise.
- If $F = P_4$, then choose $v_F$ and one vertex of $F$ as follows. If $v_F$ is adjacent to a leaf of $F$, then choose the vertex of $F$ at distance 2 from this leaf. If $v_F$ is not adjacent to a leaf of $F$, then by the claw-freeness of $G$ it is adjacent to both central vertices of $F$, and we choose one central vertex of $F$. Note that the two vertices form a 2DDS of $G[V(F) \cup \{v_F\}]$.
- If $F = C_4$, then choose $v_F$ and a vertex of $F$ so that the resulting set dominates $V(F) \cup \{v_F\}$ in $G$.
- Assume $F = G_3$. Then choose $v_F$ and two vertices of $F$: If $v_F$ is adjacent to a leaf of $F$, say $w$, then choose the two support vertices of $F$ that are not adjacent to $w$. Suppose $v_F$ is adjacent to no leaf of $F$. Let $z$ denote the central vertex of $F$ and $t$ the vertex of degree 2 in $F$. By the claw-freeness of $G$ the vertex $v_F$ is adjacent to $z$. If $v_F$ is adjacent to $t$, then choose vertex $z$ and one of the support vertices in $F$ of degree 3 in $F$. If $v_F$ is not adjacent to $t$, then by the claw-freeness of $G$ the vertex $v_F$ is adjacent to all vertices of degree 3 in $F$; in this case, choose vertices $z$ and $t$. 

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By construction the resultant set $S$ is a 2DDS of $G$. Furthermore, it is guaranteed to have cardinality at most $2n/5$, except possibly when $|T| > 2|Y|/5$. Recall that $\{u, v\} \subseteq Y$, and so $|Y| \geq 2$. Further $n \geq 3$.

Suppose that there is only one $K_1$-fragment, which necessarily consists of the vertex $u$. Then, $|T| = 1$ and we may assume that $T = \{v\}$. We are done unless $|Y| = 2$; so assume $Y = \{u, v\}$. In particular, $X_1 = \{v\}$, implying that there is no fragment $F \notin \{P_2, P_4, C_4, G_3\}$. If there is some fragment $F \in \{G_3, P_2\}$, then the set $S$ is guaranteed to have cardinality at most $2n/5$. So we may assume all remaining fragments are $P_4$- or $C_4$-fragments. By construction, then, $X \subset T$. If $|X| \geq 3$, then $S \setminus \{v\}$ is a 2DDS of cardinality less than $2n/5$ and we are done.

Hence we may assume that $|X| = 2$, implying that there is only one remaining fragment, which is either a $P_4$- or $C_4$-fragment. Hence, $n = 7$ and $P_7$ is a subgraph of $G$. But then $\gamma_d^2(G) \leq \gamma_d^2(P_7) = 2 < 2n/5$, and once again we are done.

Suppose there are at least two $K_1$-fragments. We note that $|Y| \geq 4$ and $|X_1| \geq 2$. In this case, $|T| = 2$ and we may assume that $T$ consists of the vertex $v$ and one other vertex of $X_1$. If $|Y| \geq 5$, then $|T| \leq 2|Y|/5$; so we may assume that $|Y| = 4$. In particular, there are exactly two $K_1$-fragments, one of which consists of the vertex $u$. We note that $T = X_1$ and that $|X_1| = 2$. Further there is no fragment $F \notin \{P_2, P_4, C_4, G_3\}$. By construction, if there is a remaining fragment $F \in \{P_2, P_4, C_4, G_3\}$ that is not a $P_2$-fragment with exactly one vertex adjacent to $v_F$, then the vertex $v_F \in S$. But then $S \setminus \{v\}$ is a 2DDS of cardinality at most $2n/5$ and we are done. Hence we may assume that every remaining fragment $F$ is a $P_2$-fragment with exactly one vertex adjacent to $v_F$.

If there are two such fragments, then $|S| \leq 2n/5$, and we are done. Hence we may assume that there is exactly one remaining fragment. But then $G = G_3$, completing the proof of the theorem. □

With some more work, one can characterize the graphs that achieve equality in the above bound:

**Theorem 18** If $G$ is a connected claw-free graph of order $n \geq 8$ with $\gamma_d^2(G) = 2n/5$, then $G = G_{2,t}$ for some $t \geq 1$.

We omit the proof.
4.2 Conjecture for $b \geq 3$

We pose the following conjecture.

**Conjecture 1** For $b \geq 2$, if $G$ is a connected claw-free graph of order $n \geq 4b-1$, then $\gamma_{bd}(G) \leq \left(\frac{b}{2b-1}\right) n$.

If Conjecture 1 is true, then the upper bound is sharp, by the graphs $G_{b,t}$ described earlier. Further, the restriction on the order in Conjecture 1 cannot be relaxed. For example, consider the graph $H_b$ obtained from the corona $\text{Cor}(K_b \cup K_{b-1})$ by adding a new vertex adjacent to all the vertices in $V(K_b \cup K_{b-1})$. Then, $H_b$ is a connected claw-free graph of order $4b-1$. However, $\gamma_{bd}(H_b) = 2b-1 > \left(\frac{b}{2b-1}\right) |V(H_b)|$.

5 Algorithmic Questions

5.1 Complexity

For an input graph $G$ and integer $k$, we define the problem **$b$-disjunctive domination** to be whether the graph has a $b$-disjunctive dominating set of size at most $k$.

**Theorem 19** If $b \geq 2$, then $B$-disjunctive domination is NP-complete, even when restricted to planar or bipartite graphs.

**Proof:** It is easy to check whether a given set $S$ is a $b$DDS. Thus the decision problem **$b$-disjunctive domination** is in NP.

Dominating is known to be NP-complete, even when restricted to planar or bipartite graphs (see [4]). Now we reduce Dominating to our decision problem **$b$-disjunctive domination**: Given a graph $G$, form a graph $H_G$ as follows. Subdivide $G$; let $X$ be the original vertices of $G$ and $Y$ the new vertices in the subdivision. For each vertex $y \in Y$, attach $b$ copies of $P_3$ to $y$ by their ends. For each vertex $x \in X$, attach $b-1$ copies of $P_3$ to $x$. Figure 7 shows an example.
Let $T$ be a minimum $b$DDS of $H_G$. We note that either the leaf vertex or the center vertex of each added $P_3$ must be in $T$. So if $C$ denotes the set of center vertices of the added $P_3$’s, we may assume that $C \subseteq T$ and $T$ contains no leaf. Thus, all of $Y$ is $b$-disjunctively dominated, while every vertex in $X$ must be at distance at most 2 from a vertex in $T \setminus C$.

Now, let $v \in T \setminus C$ and suppose $v \notin X$. By the minimality of $T$, there exists some $w \in X$ within distance 2 of $v$. Thus, $T \cup \{w\} - \{v\}$ is also a bDDS of $H_G$. Hence, we can assume that $T \subseteq C \cup X$. Further, $T \cap X$ dominates $G$.

Conversely, if $S$ is any dominating set of $G$, then $C \cup S$ is a bDDS of $H_G$. That is, $\gamma_b(H_G) = n(b-1) + mb + \gamma(G)$, where $n$ is the number of vertices and $m$ the number of edges of $G$. Hence we have shown that domination reduces to $b$-disjunctive domination, which completes the proof. □

Another proof of this result follows from Theorem 9, since computing the $b$-domination number is known to be NP-complete.

5.2 Tree algorithm

In this section we provide a linear-time algorithm for the $b$-disjunctive domination number of a tree for fixed $b$. We use the approach of Wimer et al. [12].

Figure 7: Example of the reduction
Their approach uses dynamic programming. Specifically, the final algorithm roots the tree at some vertex and performs a postorder traversal that calculates at each vertex $v$ a vector $(c_i)$ based on the vectors of its children. Each entry $c_i$ is determined by the subtree $T_v$ rooted at $v$—specifically it counts the minimum number of vertices in a set with some property in that subtree. (For example, the algorithm for minimum dominating set calculates at each vertex $v$, the minimum cardinality of (1) a dominating set containing $v$, (2) a dominating set not containing $v$, and (3) a set that dominates every vertex except $v$.)

Define a marked tree as a rooted tree where some subset of the vertices have been chosen/marked. Define a class as a set of marked trees. To simplify the creation of the final algorithm, Wimer et al. view the tree as being built by repeatedly taking two rooted trees $T_v$ and $T_w$ and making $w$ the child of $v$. The key steps in using this approach are:

- to identify a partition of the set of all marked trees into a finite number of classes, and
- to determine a table that, given the classes of two marked trees $T_v$ and $T_w$, tells one the class of the tree formed by making $w$ a child of $v$.

From the table there is a standard procedure to produce the algorithm (see [12]). So our task it to determine the classes and the table.

### 5.2.1 The classes

We now determine the classes. As is usual in this approach, there is a class that contains all invalid marked trees. Here invalid means that there is some descendant vertex at distance two or more away from the root that is insufficiently dominated.

For all other classes, it is possible to find a supertree with a bDDS $S$ such that $S$ restricted to the original tree is precisely the marked vertices. We will use $\otimes$ to denote the class where the root is marked; that is, the set of all marked trees where the marked vertices form a bDDS that contains the root.
If the tree $T_v$ with root $v$ has marked set $S$ such that $v \notin S$, then the class of $T_v$ is determined by three parameters $I$, $J$, and $K$:

- Let $I(v)$ be the number of children of $v$ in $S$.
- Let $J(v)$ be the number of vertices in $S$ that are at distance two away from $v$, if $v$ is dominated by none of its children, and let $J(v) = \infty$ otherwise.
- Let $K(v)$ be the minimum value of $J(x)$ among all children $x$ of $v$.

We can represent these values with a triple $(i, j, k)$, where $i = I(v)$, $j = J(v)$, and $k = K(v)$. Note that when $j$ or $k$ is less than $b$, then the marked vertices do not form a $b$DDS for that tree.

Now, we observe:

- If any of the values is at least $b$, its actual value does not matter. So we write $\infty$ to refer to any such value.
- Since every neighbor of $v$ is also at distance of two away from any other neighbor of $v$, it follows that $K(v) \geq I(v)$.
- If $K(v) \leq b - 1$, then in a $b$DDS $v$ must have a marked neighbor outside $T_v$. That is, if $k \leq b - 1$ then $k$ must eventually be increased by increasing $i$, which causes the value of $j$ to become $\infty$. For this reason, we may assume that if $k \leq b - 1$ then $j = \infty$.

Thus there are exactly $(b^2 + 5b + 4)/2$ valid classes.

### 5.2.2 The table

Now, we can construct the table for the algorithm using these classes. That is, given $T_v$ in class $(i, j, k)$ and $T_w$ in class $(i', j', k')$, the table tells one what new class $T_v$ will be in after we make $v$ the parent of $w$. Here is the summary table.

<table>
<thead>
<tr>
<th>$T_v \setminus T_w$</th>
<th>$\otimes$</th>
<th>$0, j', k' \leq b - 2$</th>
<th>$0, j', b - 1$</th>
<th>$i', j', \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\otimes$</td>
<td>$\otimes$</td>
<td>$-$</td>
<td>$\otimes$</td>
<td>$\otimes$</td>
</tr>
<tr>
<td>$i, j, k$</td>
<td>$i + 1, \infty, k + 1$</td>
<td>$-$</td>
<td>$-$</td>
<td>$i, j + i', \min(k, j' + i)$</td>
</tr>
</tbody>
</table>
where − means the result is invalid.

The table can be justified as follows. Assume first that $T_w$ is in class $\otimes$. This means that $v$ has another child that is marked; hence both $i$ and $k$ increase. Assume second that $w$ is not marked. The value for $k'$ increases by at most 1 when the $vw$ edge is added, and even then, only if $v$ is in the bDDS. Hence, if $k' \leq b - 2$, then no such set is a bDDS. Also, if $k' = b - 1$, then $T_v$ can only be case $\otimes$. Further, when the $vw$ edge is added, $w$ becomes a neighbor of $v$, so it is considered for the new $K(v)$. Hence, $k_{new} = \min(k, j' + i)$. Also, the neighbors of $w$ are at distance two away from $v$. Hence, $j_{new} = j + j'$.

Finally, from the table, a linear-time algorithm can be constructed to calculate the $b$-disjunctive domination number of any tree.

5.2.3 The special case of $b = 2$

We include the specific table in the case where $b = 2$. By the above discussion, there are the following 9 classes:

1 : $\otimes$
2 : $(0, \infty, 0)$
3 : $(0, \infty, 1)$
4 : $(1, \infty, 1)$
5 : $(0, 0, \infty)$
6 : $(0, 1, \infty)$
7 : $(0, \infty, \infty)$
8 : $(1, \infty, \infty)$
9 : $(\infty, \infty, \infty)$

We can now form a Wimer table, given in Figure 8. In each case, a tree (of smallest order) in the class is given.
Figure 8: The Wimer table for 2-disjunctive domination.
6 Conclusion and open problems

In this paper we introduced disjunctive domination and provided bounds on the parameter, especially for trees and claw-free graphs. The relationship with standard domination proved useful. We showed there is a linear-time algorithm for calculating $\gamma_d^b$ for trees, and that computing $\gamma_d^b$ in general is NP-complete. Open problems include finding $\gamma_d^b$ for regular graphs, specifically for cubic graphs and regular products such as the hypercube.

References


