Summary of West 5.1: Vertex coloring and upper bounds

A $k$-coloring is a labeling of the vertices with labels/colors from a universe of size $k$. A coloring is proper if every pair of adjacent vertices receive different colors. The chromatic number $\chi$ is the least $k$ such that the graph is $k$-colorable. Example: bipartite = 2-colorable; Petersen is 3-colorable.

Lower bound: clique number $\omega$; $n/\alpha$ (cannot use one color more than independence number times). Upper bound: $1 + \Delta$ (trivial greedy)

The cartesian product: vertex set is all pairs $(u, v)$ of $V(G) \times V(H)$ with pairs adjacent if the same in one coordinate and adjacent in the other. For example: hypercube $Q_k = Q_{k-1} \square K_2$. Cute theorem: $\chi(G \square H) = \max(\chi(G), \chi(H))$.

Interval graphs: A collection of intervals that are adjacent if they overlap. Theorem: chromatic number of interval graph is the clique number. Proof: Run greedy algorithm on intervals ordered by left end-point. If need color $k$ for that interval, it overlaps intervals with colors 1 through $k - 1$, but it and they must form a clique.

Brooks’ Theorem: Except for clique and odd cycle, $\chi \leq \Delta$. Proof idea: do a “greedy” coloring but ensure when get to last vertex it has two neighbors of the same color.

A graph is $k$-critical if the chromatic number is $k$ but decreases on the removal of any edge. Minimum degree of a $k$-critical graph is at least $k - 1$. (Szekeres–Wilf) $\chi$ is at most 1 plus maximum $\delta(H)$ over all subgraphs $H$. 
Summary of West 5.2: Structure of \( k \)-chromatic graphs

Mycielski construction: Can obtain triangle-free graphs with arbitrarily large chromatic number. The idea is to start with \( G_2 = K_2 \) and construct \( G_{i+1} \) from \( G_i \) by duplicating all the vertices into an independent set, and adding a final vertex. \( G_4 \) is known as the Grötzsch graph.

A complete multipartite graph is simple graph where the vertex set can be partitioned into subsets such that each subset is an independent set and any two vertices in different subsets are adjacent.

Turán’s theorem: the maximum number of edges in a graph which has no \( K_{k+1} \) is achieved by taking a complete \( k \)-partite graph with as equal parts as possible.

Theorem (Dirac): Every graph with chromatic number at least 4 contains a subdivision of \( K_4 \).