Summary of West 3.1: Matchings and Covers

A matching is a set of edges no two of which touch. A vertex is saturated if it is the end of an edge in the matching. A perfect matching saturates all the vertices. For example, $K_{2n}$ has $(2n)! / 2^n n!$ perfect matchings.

Note that a maximal matching is not the same as a maximum matching. Define symmetric difference $G \triangle H$ as the subgraph with the edges that are in precisely one of the graphs. Observation: if $G$ and $H$ are matchings, then $G \triangle H$ consists of paths and even cycles.

Given matching $M$, an $M$-alternating path is one where the edges alternate between in and out of $M$. An $M$-augmenting path is a nontrivial $M$-alternating path that starts and ends at an $M$-unsaturated vertex.

Berge: A matching $M$ is maximum if and only if there is no $M$-augmenting path. To prove existence of augmenting path if $M$ not maximum, consider union or symmetric difference of $M$ and the actual maximum matching $M'$; consider a component where $M'$ is in the majority.

Hall: In a bipartite graph with one side $X$: there is a matching that saturates $X$ if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$. TONCAS. Proof of sufficiency: consider an unsaturated vertex $u$ and let $S$ be the vertices of $X$ reachable from $u$ via $M$-alternating paths. If there is no $M$-augmenting path, then $|N(S)| = |S| - 1$. Corollary: a regular bipartite graph can be decomposed into perfect matchings.

A vertex cover is a set of vertices intersecting all edges. A set is a vertex cover if and only if its complement is an independent set. An edge cover is a set of edges saturating all vertices. Notation: $\alpha$ is maximum size of independent set, $\beta$ is minimum size of vertex cover, $\alpha'$ is maximum size of matching, $\beta'$ is minimum size of edge cover.

König–Egerváry: in a bipartite graph, maximum matching equals minimum vertex cover. Inequality always. (Proof of equality: show that a minimum vertex cover restricted to each side obeys Hall.)

Gallai: $\alpha' + \beta' = n$ for graph without isolates.
Summary of West 3.2,3.3: Algorithms and applications, matchings in general graphs

Berge’s theorem gives an algorithm to find a maximum matching in a bipartite graph. Starting at an unmatched vertex, one determines the set of vertices reachable by $M$-alternating paths, and thus either finds an $M$-augmenting path or proves that none exists. The Hungarian method [omitted] finds a matching of maximum weight.

Tutte’s 1-factor theorem says that a graph has a perfect matching if and only if $o(G - S) \leq |S|$ for all sets of vertices $S$, where $o(G - S)$ is the number of odd components of $G - S$. [Proof omitted]

Petersen’s theorem says that any cubic (3-regular) graph without a cut-edge has a perfect matching.