A linear map $T$ from one vector space $V$ to another $W$ is a map that preserves the sum and the scalar product: $T(u + v) = Tu + Tv$ and $T( av) = a(Tv)$. For example, the zero map $0$ maps everything to zero; the identity map $I$ maps everything to itself. Other examples include differentiation of polynomials and backward shift of sequences.

A linear map is completely determined by what it does to a basis of $V$. The set $\mathcal{L}(V, W)$ of all linear maps is itself a vector space. Multiplication of maps means composition: $(ST)(v) = S(Tv)$. It has some nice properties.

The null space or kernel of a map, denoted null $T$, is the set of all elements that get mapped to 0. It is a subspace of $V$. The range of a map is the set of all elements that get mapped to. It is a subspace of $W$. A map is injective (one-to-one) iff the null space is $\{0\}$. A map is surjective (onto) iff the range is all of $W$.

If $T$ is a linear map from (finite-dimensional) $V$ to $W$, then

$$\dim V = \dim \text{null } T + \dim \text{range } T.$$  

The proof idea is to extend a basis of the null space of $T$ to a basis of $V$ and show that the image of the extra vectors is a basis of the range.

Consequently, for example, if $V$ has bigger dimension then $W$, then no map is injective; if $W$ has bigger dimension then $V$, then no map is surjective. The solution set of a system of homogeneous linear equations is a vector space.