9 Discrete Dynamical Systems

9.1 Overview

We consider a deterministic (that is, no randomness) dynamical system but with discrete time. These could be successive populations, for example, cicadas. As a mathematical formulation, we have a function $g(x)$ such that the population $x_n$ at time $n$ is given by the recurrence

$$x_{n+1} = g(x_n)$$

That is, there is a sequence $x_0, g(x_0), g^2(x_0), g^3(x_0), \ldots$

9.2 Discrete Growth

One of the earliest examples of this was examined by the economist Malthus:

$$x_{n+1} = rx_n$$

where $r$ is the reproductive factor. This recurrence easily solves to $Ar^n$ where $A = x_0$. This exponential growth led to dire predictions.

A more reasonable model where resources are limited is to say that growth slows as the population grows. For example, the Beverton-Holt model is the dynamical system

$$x_{n+1} = \frac{rx_n}{1 + ax_n}$$

This recurrence has the solution

$$x_n = \frac{Kx_0}{x_0 + (K - x_0)r^{-n}}$$

where $K = (r - 1)/a$. This is called the carrying capacity since, $x_n$ is asymptotic to $K$ (assuming $r > 1$).

9.3 The Logistics Model

The logistics model represents the situation where reproduction proportional to population, and starving/competition proportional to the number of interactions
of the population, which is proportional to the square of population. Thus, for some $r$ and $s$, we have $x_{n+1} = rx_n - sx_n^2$. To simplify the discussion, we can scale, replacing $x_n$ by $rx_n/s$, and the recurrence simplifies to

$$x_{n+1} = rx_n(1 - x_n).$$

The big result is that the behavior varies considerably as $r$ changes.

### 9.4 Fixed Points

The solutions to the equation $g(x) = x$ are called **fixed points**. If the population $x_0$ is a fixed point then it remains constant. However, otherwise there are several possible behaviors. The simplest are that the fixed point attracts or repels.

Mathematically: We say $x_0$ is an **attracting point** if there exists $\varepsilon > 0$ such that $\forall x \in [x_0 - \varepsilon, x_0 + \varepsilon]$ it holds that $\lim_{n \to \infty} f^{(n)}(x) = x_0$. The **basin of attraction** is the set of points that converge to $x_0$. Conversely: we say $x_0$ is a **repelling point** if there exists $\varepsilon > 0$ such that $\forall x \in [x_0 - \varepsilon, x_0 + \varepsilon]$ it holds that there exists $N$ such that $f^{(n)}(x) \not\in [x_0 - \varepsilon, x_0 + \varepsilon]$ for all $n \geq N$.

**Example:** Consider the logistics map $g(x) = 2x(1 - x)$. Note that $x = \frac{1}{2}$ is a fixed point. Computer code shows that if one starts near $x = \frac{1}{2}$, then process will tend toward $\frac{1}{2}$. In contrast, $x = 0$ is also a fixed point, but a population that starts near 0 moves away. That is, $\frac{1}{2}$ is an attracting point and 0 a repelling point.

**Example.** Consider the following map $g(x) = 4x^2(1 - x)$. Here there are two fixed points, $x = 0$ and $x = \frac{1}{2}$ (coincidence). Computer code shows that if one starts between 0 and $\frac{1}{2}$, then system tends to 0: this is an attracting point. However, if one starts a little above $\frac{1}{2}$ then converges to it.

### References

Wikipedia.
