22 Linearization and Stability

One can determine the behavior close to the equilibrium point. This is similar to linear approximation or differentials. One implementation of the idea is to translate the system so that the origin is the equilibrium point. Then when one looks at the right-hand-sides, one discards all terms that are quadratic or higher, on the grounds that they are insignificant.

22.1 Linearization

Recall that we had the system
\[
\dot{u} = u(1-v) \quad \dot{v} = \alpha v(u - 1).
\]
So let \( U = u - 1 \) and \( V = v - 1 \). Then we have \( \dot{U} = \dot{u} = u(1-v) = (U + 1)(-V) = -V - UV \approx -V \). Similarly, we have \( \dot{V} = \dot{v} = \alpha v(u - 1) = \alpha (V + 1)U \approx \alpha U \).
That is,
\[
\dot{U} = -V \quad \text{and} \quad \dot{V} = \alpha U.
\]

This system we can solve. If we plug the second into the first, we get that \( \ddot{U} = -\alpha U \) (the equation for simple harmonic motion), and thus \( U \) is given by a linear combination of \( \cos(\sqrt{\alpha}t) \) and \( \sin(\sqrt{\alpha}t) \). That is, the solution to \( U \) follows an ellipse.

In general, in the linearized translated system, each derivative is expressed as a linear combination of the variables. That is, we have the vector equation
\[
\dot{x} = Ax.
\]
(This is related to the Jacobian.)

Now, take as “ansatz” that the solution is given by \( x = v e^{\lambda t} \). Then \( \dot{x} = \lambda v e^{\lambda t} \).
This means that the above equation requires that
\[
A v = \lambda v.
\]
That is, \( v \) is eigenvector for eigenvalue \( \lambda \).

In our example above, we had \( A = \begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix} \). This has characteristic equation \( \lambda^2 + \alpha = 0 \) and so \( \lambda = \pm \sqrt{\alpha} i \). That is, the eigenvalues are purely imaginary.
22.2 Classification

Even if we restrict to the two-variable case, a lot can happen. (Recall that the eigenvalue are solutions to a quadratic equation; thus they are either both real or neither real.) In the following list we ignore the possibility that the two eigenvalues are equal:

<table>
<thead>
<tr>
<th>(distinct) eigenvalues</th>
<th>behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>both positive</td>
<td>repeller</td>
</tr>
<tr>
<td>both negative</td>
<td>attractor</td>
</tr>
<tr>
<td>opposite signs</td>
<td>saddle</td>
</tr>
<tr>
<td>complex with $-\text{ve}$ real</td>
<td>inward spiral</td>
</tr>
<tr>
<td>complex with $+\text{ve}$ real</td>
<td>outward spiral</td>
</tr>
<tr>
<td>purely imaginary</td>
<td>“center”</td>
</tr>
</tbody>
</table>

These claims can be justified. Consider for example the case that we have two real eigenvalues $\lambda_1$ and $\lambda_2$. That is, the solution $\mathbf{x}$ is a linear combination of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$. If both eigenvalues are negative, then the solution tends to 0. If both are positive, then solution tends to $\pm \infty$. One can get further insight by a change of coordinates that diagonalizes the system; this will mean the one variable is given by $Ae^{\lambda_1 t}$ and the other by $Be^{\lambda_2 t}$ and so they are related by $y = \pm Cx^{\lambda_2/\lambda_1}$. This explains the saddlepoint.

For systems with more variables, more can happen. E.g. the chaotic three-dimensional Lorenz system.

22.3 Interpretation

Our first equilibrium point is $(0, 0)$. This has the linearized system $\dot{u} = u$ and $\dot{v} = -\alpha v$, which has eigenvalues 1 and $-\alpha$. By the above, we have a saddle. In real-life the only way the system can go to the $(0, 0)$ is for the prey to be exterminated (e.g. by human intervention). If the predators die out, then the prey will just multiply.

Our second equilibrium point is $(1, 1)$. We have already seen that the solution follows an ellipse: that is, the solution cycles. The table calls this situation a
“center”. The point is that we have a cycle, but the solution is unstable: any perturbation of the original will introduce a real part to the eigenvalues, and the solution will then become a spiral inward or outward.

22.4 Other Model Inferences

The original motivation for Volterra was the effect of the temporary cessation of fishing during WWI. It was noted that the prey (the species that the humans had been catching) declined. If the model holds, then what happens is that human fishing can be added by increasing mortality/decreasing growth. When human fishing stops, the equilibrium point moves. The system then cycles around that equilibrium point. When fishing resumes, the system is on a different level curve. Volterra discussed this.

References

Wikipedia.