15 Dimensional Analysis

Dimensional analysis is the idea that the formula must give the correct physical units.

15.1 Classic pendulum

Consider a classic pendulum: a blob of mass $M$ on a string of length $\ell$ acting under gravity. In a mechanics course we would proceed something like this. We have $F = Ma$ where $F$ is the force and $a$ the acceleration. The blob moves along a circle. The force that matters is the force tangential to the circle. So if $\theta$ is the angle from vertical, this force is $-Mg \sin \theta$. On the other hand, the acceleration along the circle is equal to the angular acceleration times the length of the string. Thus, with the masses canceling, we get

$$\ell \ddot{\theta} = -g \sin \theta,$$

where we use a dot to indicate derivative with respect to time.

Now even that ODE we cannot solve. So we linearize: we say that for small $\theta$, $\sin \theta \approx \theta$. We can solve

$$\ddot{\theta} = -\frac{g}{\ell} \theta.$$

This has solution $\theta = A \cos(\omega t) + B \sin(\omega t)$ where $\omega = \sqrt{g/\ell}$. In particular, the period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{\ell}{g}}.$$

15.2 Classic pendulum: dimensional analysis

Assume the period $T$ is a function of $M$, $\ell$, $g$, and $\theta^*$, the maximum angle. Express everything in SI units (the metric system); this is not essential, but it simplifies conversions. We have

$$M = kg \quad \ell = m \quad g = m \ s^{-2} \quad \theta^* = \text{dimensionless} \quad \text{and} \quad T = s.$$

So we will get no information about the dependence on $\theta^*$. So we will just:

**Try** $T = f(\theta^*) M^a \ell^b g^\gamma$. 
Note that the RHS has units $(kg)^\alpha (m)^{\beta + \gamma} (s)^{-2\gamma}$. For the RHS to have the right units, we need
\[ \alpha = 0, \quad \beta + \gamma = 0, \quad \text{and} \quad -2\gamma = -1. \]
This solves to \( \alpha = 0, \beta = \frac{1}{2} \) and \( \gamma = -\frac{1}{2} \). That is,
\[ T = f(\theta^*) \sqrt{\ell/g} \]

**References**
