12 The Erdős–Renyi Random Graph

12.1 Model

Consider producing a graph by the following process. Choose \( n \) the number of vertices and pick a real number \( p \) between 0 and 1. The random graph \( G(n, p) \) is obtained by tossing an independent coin for each of the \( \binom{n}{2} \) possible edges such that the edge is in with probability \( p \). Entire books have been written on this graph (or graph distribution). This graph was introduced by Erdős to provide examples for Ramsey theory (see later). The graph was later studied in great detail. Sage calls this \texttt{RandomGNP}. Here is a picture of the case of \( G(15, 0.33) \).

12.2 Properties

Some simpler observations. The degree of a specific vertex (how many neighbors it has) has the binomial distribution \( B(n, p) \). In particular, the degrees are concentrated around \( np \) (e.g. central limit theorem or Chernoff bounds). But real-world graphs have a much wider spread of degrees.

Another observation. The distance between two vertices is defined as the number of edges on the shortest path between them, and the diameter of a graph is defined as the maximum distance between any two vertices. Consider
any two vertices \( u \) and \( v \). The probability that another vertex \( w \) is a common neighbor is \( p^2 \). So the probability \( u \) and \( v \) do not have a common neighbor is \((1 - p^2)^{n-2}\), which is exponentially small provided \( p \) is a constant. Even the diameter is two “almost surely” (meaning the probability tends to 1).

12.3 More properties

In what follows we fix \( p = \frac{1}{2} \). It does not change the essential nature of anything that follows, but simplifies the discussion slightly.

The graph contains all small subgraphs. For example, for any fixed graph \( H \) appears in \( G(n, \frac{1}{2}) \). The random graph is also sometimes called \textit{existentially closed}. Fix some integer \( k \). For every sets \( A, B \) of vertices of size \( k \) almost surely there exists a vertex that is adjacent to all of \( A \) but none of \( B \). Again, the probability that a vertex doesn’t do the job is exponentially small (assuming \( k \) is a constant).

The \textit{connectivity} of a graph is defined as the minimum number of vertices whose removal disconnects the graph. (Note that some of the applied sciences use the term connectivity of a vertex, which mathematicians call its degree.) The connectivity of \( G(n, \frac{1}{2}) \) is almost surely \( n/2 - o(n) \). Mathematical notation: \( o(n) \) means a function \( f(n) \) smaller than \( n \); that is, \( f(n)/n \) tends to 0. The proof idea is that if one removes a set smaller than this, almost surely every pair of vertices still have a common neighbor.

Consider the \textit{eigenvalues} of the graph \( G(n, \frac{1}{2}) \). Meaning, consider the \textit{adjacency matrix} of the graph: that is, a matrix \( A \) such that the entry \( a_{ij} \) is 1 if vertices \( i \) and \( j \) are joined and 0 otherwise. This is a real symmetric matrix; therefore, it has \( n \) real eigenvalues. It can be shown that the dominant eigenvalue is \( n/2 + o(n) \). Further, all other eigenvalues are \( o(n) \).

References