11 Discrete Dynamics: Attractors and Chaos

11.1 Periodic Attractors

Consider now the logistics map with \( r \) a little bigger than 3. In this case, the sequence seems to converge towards a period-2 oscillation. For example, here is a cobweb diagram for \( r = 3.3 \).

But maybe that’s just coincidence. Well, what period-2 oscillation means is that the subsequence \( x_0, x_2, x_4, x_6 \) is converging. So we should look at the recurrence \( g(g(x)) \). That function has a fixed point at

\[
p = \frac{r + 1 \pm \sqrt{(r + 1)(r - 3)}}{2r}
\]

And we can use the earlier theorem. We need to calculate the derivative of \( g^{(2)}(x) \). By the chain rule we can calculate it, and using a computer, find that its absolute value is less than 1 until \( r \approx 3.4495 \). So we call this pair (the two values of \( p \) from above) a periodic attractor.

Actually, note that by the chain rule, if we have a periodic attractor \( p_1, \ldots, p_k \), then \( g^{(k)}(p_1) = g'(p_1) \cdot g'(p_2) \cdots g'(p_k) \).

For \( 3.44949 < r < 3.56995 \), the process (almost always) settles to an oscillation, and the period of the oscillation is some power of 2.
11.2 Chaos

For the range $4 > r > 3.56995$ the logistic map is often chaotic. That is, the system does not settle down to an oscillation, and tiny changes in initial conditions lead to considerable deviation in behaviour.

Now consider significant value around 3.84. Here a period-3 oscillation emerges. Famous theorem says: possibility of odd-period implies chaos.

We can represent the range of behavior for each $r$ on a bifurcation diagram.

References

Wikipedia.
