10 Convergence and Cobwebs

10.1 Fixed Points Again

We saw earlier some examples of behavior with regard to fixed points. One can put mathematics to it:

**Theorem 1** Assume the recurrence function $g(x)$ is differentiable at fixed point $p$.

(i) If $|g'(p)| < 1$, then $p$ is attractive.

(ii) If $|g'(p)| > 1$, then $p$ is repelling.

**Proof.** (i) By the definition of derivative, for any real $a$ with $|g'(p)| < a < 1$, there is an $\varepsilon$ such that

$$\frac{|g(x) - g(p)|}{|x - p|} < a$$

for all $x$ in $(p - \varepsilon, p + \varepsilon)$. Now, since $g(p) = p$, this means that $|g(x) - p| < a|x - p|$; that is, $g(x)$ is closer to $p$ than $x$ was. In particular, it follows that $|g^{(n)}(x) - p| < a^n|x - p|$. That is, $g^{(n)}(x)$ converges to $p$.

(ii) This is the same idea. Proof left as an exercise. (Except note that definition we gave before is incorrect.)

**Example.** We saw previously the logistics map $g(x) = 2x(1 - x)$ had fixed points $0$ and $\frac{1}{2}$. Note that $g'(x) = 2 - 4x$, so that $g'(0) = 2$ and $g'(\frac{1}{2}) = 0$. So $0$ is repelling and $\frac{1}{2}$ is attracting.

**Example.** We saw previously that the map $g(x) = 4x^2(1 - x)$ also had fixed points $0$ and $\frac{1}{2}$. Here $g'(x) = 8x - 12x^2$, so that $g'(0) = 0$ and $g'(\frac{1}{2}) = 1$. So $0$ is attracting. The theorem says nothing about $\frac{1}{2}$; we saw that it is a mixture.

**Example.** Consider the general logistics map again.

$$x_{n+1} = rx_n(1 - x_n)$$

This has fixed points $0$ and $(r - 1)/r$. Note that $g'(x) = r(1 - 2x)$.

It is no surprise that when $r < 1$ the population dies out. This is confirmed by noting that $g'(0) = r$, so that $x = 0$ is attractive for $r < 1$ and repelling for $r > 1$. The behavior noted for $r = 2$ above continues in general for $1 < r < 3$, since $g'((r - 1)/r) = 2 - r$. 

10.2 Cobweb Diagrams

One can obtain some visual depiction of the behavior by using a cobweb diagram. This joins the points \((x_0, x_1)\) to \((x_1, x_1)\) to \((x_1, x_2)\) and so on; it is a curve that goes horizontal and vertical, bouncing off alternately the curves \(y = g(x)\) and \(y = x\).

Example. Here are cobweb diagrams for the two functions discussed above.

![Cobweb Diagrams](image)

10.3 Aside: The Cobweb Theorem in Economics

A related theorem occurs in economics. This shows that: whether prices converge for a seasonal product (such as agriculture) depends on the slopes of the demand and production curves (that is, their elasticities). See the Wikipedia article.

References

Wikipedia.
