Assignment 5

(This is an individual assignment)

1. (a) Compute $2^{38} \mod 7$.
   (b) Compute $3^{29} \mod 20$.
   (c) Compute $5^{33} \mod 13$.

2. Prove by induction that for all $n \geq 1$
   \[ \sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2} \]
   (Note that the sum starts $-1 + 4 - 9 + 16 \ldots$)

3. Prove by induction that for all $n \geq 1$:
   \[ \sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1} \]

4. Prove by induction that $3^n \geq n^3$ for all integers $n \geq 3$.

5. Consider the following game. Some number $n$ is chosen, and on a piece of paper is written down all its factors (including itself and 1). The two players take turns naming a factor that has not been crossed out. Each time they name a factor, that factor and all multiples of it are crossed out. The player who has to take the factor 1 loses.
   For example, suppose $n = 24$. The factors are 1, 2, 3, 4, 6, 8, 12, and 24. Say the first player names 4. Then the remaining factors are 1, 2, 3, and 6. Say the second player names 3. Then, the remaining factors are 1 and 2. The first player clearly names the 2, and the second player loses.

   (a) Show that the FIRST player can always win if $n$ is prime or a power of a prime.
   (b) Show that the FIRST player can always win if $n$ is a product of two distinct primes.
   (c) Show that the FIRST player can always win if $n$ is a product of three distinct primes.
   (d) What should happen for $n = 24$ if both players play optimally?

Due: Start of class Thursday 19 October

Game of the Week. At the end of the TV game-show Jeopardy, the players encounter Final Jeopardy. There they wager some of their winnings so far on answering the Final Jeopardy question, after which only the leader gets to keep their winnings and return the next day. For example, maybe Alejandro has $10,000, Brianna has $7,000, and Carter has $3,000. How should they wager?