A combinatorial game is where two players alternate but there is no randomness or hidden information. The most common example is Tictactoe, also called Noughts and Crosses. But this genre also include Chess, Checkers, Go, and Nim.

E4.1 Game Trees

There is a natural way to represent the game in what resembles a family tree. The nodes of the tree represent board positions. For each node, the children are the next positions. The game ends with certain positions being reached. Since this is a tree, these positions are called leaves. This is called the game tree.

If the tree is small enough, we can draw it and proceed as follows. We move from the leaves upwards. We start by marking all final positions as to whether the first player wins or loses. We then consider a position one move from the end. If every move that a player can make causes him to lose, then that position is a loss. On the other hand, if some move causes her to win, then that position can be viewed as a win. Assuming perfect play, we can then label each position one move from the end. And repeat the same argument with positions two moves from the end, and so up the tree.

To simplify the writing, we view the game from the first player’s perspective. So we label a node as a Win if the first player wins, and as a Loss if the first player loses. Here is a simple tree, and the resultant labeling.

Note that this process assigns a label to the root node of the tree. If a draw is impossible, then the root node is labeled as either first player win or second player win. This means that the game has a definite value/result:
**Theorem.** Either there is a strategy that guarantees the first player a win, or there is a strategy that guarantees the second player a win.

We can also generalize this labeling to handle games where there are more than two outcomes (for example win, loss and draw). Thus in chess there exists one of the following: (a) a strategy that guarantees the first player wins; (b) a strategy that guarantees the second player wins; or (c) a pair of strategies that guarantee that neither player can lose (and so the game ends in a draw). Though we don’t know which of the three cases applies, as the tree is rather large.

### E4.2 Symmetry and Strategy Stealing

Consider the following game:

**Dawson’s Kayles.** Some number of dots are drawn on a line. Two players alternate by picking two consecutive unused dots and joining them. The last person to make a move is the winner.

The picture below gives the game tree for the game with 6 dots (to fit in the page, we can assume that the first person moves in the left half).

In this tree, the first player can force a win by taking the two middle dots. In all other cases, the second player can win.
It turns out that Dawson’s Kayles with an even number of dots is always a win for the first player. This is a symmetry argument. The starting move is to pair off the two central dots. After that the first player mirrors whatever the second player does but in the other half of the picture. Thus the first player is guaranteed to win.

In another direction we have what some call achievement games. In these games there is a race to see who can first accomplish something; tictactoe is an example. Most people “know” that playing first is best in Tictactoe: for much the same reason, achievement games in general favor the first player.

There is a simple argument that, under certain conditions, achievement games favor the first player. Specifically consider an achievement game where the players alternate selecting from some set. All elements of the set are available at the start. The moves and the goal are the same for both players: the first player to accumulate a given subset is the winner. (This is the case in Tictactoe.)

The claim is the first player cannot lose. Well, suppose the second player has a winning strategy. Then all the first player needs to do is follow the rules for the second player. This is called strategy stealing.

It is a different story if each move changes the set of available moves. And indeed in chess and many other board games, it is clear that there are positions where the next player to move loses.